Dielectric Rectangular Waveguide and Directional Coupler for Integrated Optics

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We study the transmission properties of a guide consisting of a dielectric rod with rectangular cross section, surrounded by several dielectrics of smaller refractive indices. This guide is suitable for integrated optical circuitry because of its size, single-mode operation, mechanical stability, simplicity, and precise construction.

After making some simplifying assumptions, we solve Maxwell's equations in closed form and find, that, because of total internal reflection, the guide supports two types of hybrid modes which are essentially of the TEM *kind polarized at right angles. Their attenuations are comparable to that of a plane wave traveling in the material of which the rod is made.*

If the refractive indexes are chosen properly, the guide can support only the fundamental modes of each family with any aspect ratio of the guide cross section. By adding thin lossy layers, the guide presents higher loss to one of those modes. As an alternative, the guide can be made to support only one of the modes if part of the surrounding dielectrics is made a low impedance medium.

Finally, we determine the coupling between parallel guiding rods of slightly different sizes and dielectrics; at wavelengths around one micron, 3-dB directional couplers, a few hundred microns long, can be achieved with separations of the guides about the same as their widths (a few microns).

I. INTRODUCTION

Proposals have been made for dielectric waveguides capable of guiding beams in integrated optical circuits very much as waveguides and coaxials are used for microwave circuitry.[1–3] Figure 1 shows the basic geometries for these waveguides. The guide is a dielectric rod of refractive index n immersed in another dielectric of slightly smaller refractive index $n(1 - \triangle)$; both are in contact with a third dielectric which may be air (Fig. 1a) or a dielectric of refractive index $n(1 - \triangle)$,

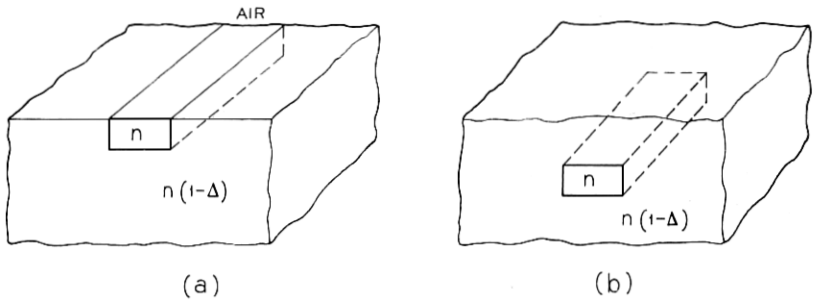


Fig. 1 — Dielectric waveguides for integrated optical circuitry.

(Fig. 1b). These geometries are attractive not only because of simplicity, precision of construction, and mechanical stability, but also because by choosing Δ small enough, single-mode operation can be achieved with transverse dimensions of the guide large compared with the free space wavelengths, thus relaxing the tolerance requirements.

Even though in a real guide the cross section of the guiding rod is not exactly rectangular and the boundaries between dielectrics are not sharply defined, as in Fig. 1, it is worth finding the characteristics of the modes in the idealized structure and the requirements to make it a single-mode waveguide.

Furthermore, directional couplers made by bringing two of those guides close together, Fig. 2, may become important circuit components.[1,2] In this paper we study the transmission through such a coupler; the modes in a single guide result as a particular case, when the separation between the two guides is so large that the coupling is negligible. Through use of a perturbation technique, we also find the coupler properties when the two guides are slightly different.

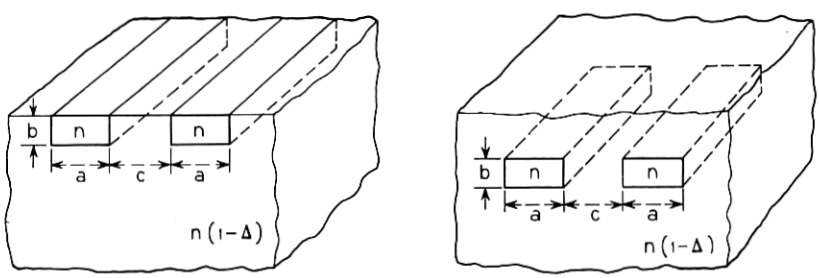


Fig. 2 — Directional couplers.

The guiding properties of the rectangular cross section guide immersed in a single dielectric are compared with those derived through computer calculations by Goell.[4] Similarly, the coupling properties of two guides of square cross section immersed in a single dielectric are compared with those of two guides of circular cross section derived by Jones and by Bracey and others.[5,6] In both comparisons agreement is quite good.

II. FORMULATION OF THE BOUNDARY VALUE PROBLEM

For analysis, we redraw in Fig. 3 the cross section of the coupler subdivided in many areas. Nine of the areas have refractive indexes n_1 to n_5; we do not specify the refractive indexes in the six shaded areas. The reasons for these choices will become obvious.

A rigorous solution to this boundary value problem requires a computer;[4,7] nevertheless, it is possible to introduce a drastic simplification which enables one to get a closed form solution. This simplification arises from observing that, for well-guided modes, the field decays exponentially in regions 2, 3, 4, and 5; therefore, most of the power travels in regions 1, a small part travels in regions 2, 3, 4, and 5, and even less travels in the six shaded areas. Consequently, only a small error should be introduced into the calculation of fields in regions 1 if one does not properly match the fields along the edges of the shaded areas.

The matching made only along the four sides of regions 1 can be achieved assuming simple field distribution. Thus the field components in regions 1 vary sinusoidally in the x and y direction; those in 2 and 4 vary sinusoidally along x and exponentially along y; and those in regions 3 and 5 vary sinusoidally along y and exponentially along x. The propagation constants k_{x1}, k_{x2}, and k_{x4} along x in media 1, 2, and

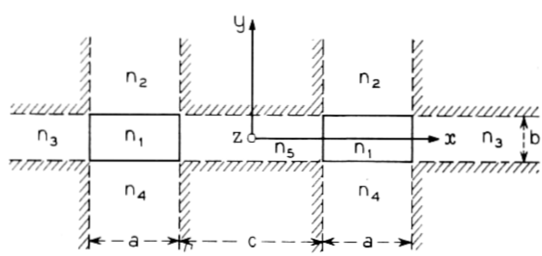


Fig. 3 — Coupler cross section subdivided for analysis.

4 are identical and independent of y. Similarly, the propagation constants k_{y1}, k_{y3}, and k_{y5} along y in the regions 1, 3, and 5 are also identical and independent of x.

In the appendix we calculate these propagation constants and find, as expected, that all the modes are hybrid and that guidance occurs because of total internal reflection. Nevertheless, because of another approximation which consists of choosing the refractive indexes n_2, n_3, n_4, and n_5 slightly smaller than n_1, total internal reflection occurs only when the plane wavelets that make a mode impinge on the interfaces at grazing angles.* Consequently, the largest field components are perpendicular to the axis of propagation; the modes are essentially of the TEM kind and can be grouped in two families, E^x_{pq} and E^y_{pq}. The main field components of the members of the first family are E_x and H_y, while those of the second are E_y and H_x. The subindex p and q indicate the number of extrema of the electric or magnetic field in the x and y directions, respectively. Naturally, E^x_{11} and E^y_{11} are the fundamental modes; we concentrate on them as we discuss the transmission properties of different structures.

III. GUIDE IMMERSED IN SEVERAL DIELECTRICS

The guide immersed in several dielectrics (Fig. 4a) is derived from Fig. 3 by choosing

$$c = \infty. \tag{1}$$

It supports a discrete number of guided modes which we group in two families E^x_{pq} and E^y_{pq} plus a continuum of unguided modes.[8,9]

3.1 *The E^y_{pq} Modes*

The main transverse field components of the E^y_{pq} modes are E_y and H_x. They are depicted in solid and broken lines, respectively, in Fig. 4a for the fundamental mode E^y_{11}. Within the guiding rod each component varies sinusoidally both along x and along y. Outside the guide each component decays exponentially. Such functional dependence is given in equation (38) and depicted in Fig. 4b. We assume $n_2 \neq n_3 \neq n_4 \neq n_5$; consequently the field distributions are not symmetric with respect to the planes $x = 0$ and $y = 0$. In Fig. 5a we assume $n_2 = n_4$ and $n_3 = n_5$; the E^y_{pq} modes depicted are either symmetric or antisymmetric with respect to the same planes. These modes look similar to those in laser

* This approximation is not very demanding. Even when n_1 is 50 percent larger than n_2, n_3, n_4, and n_5, the results are valid.

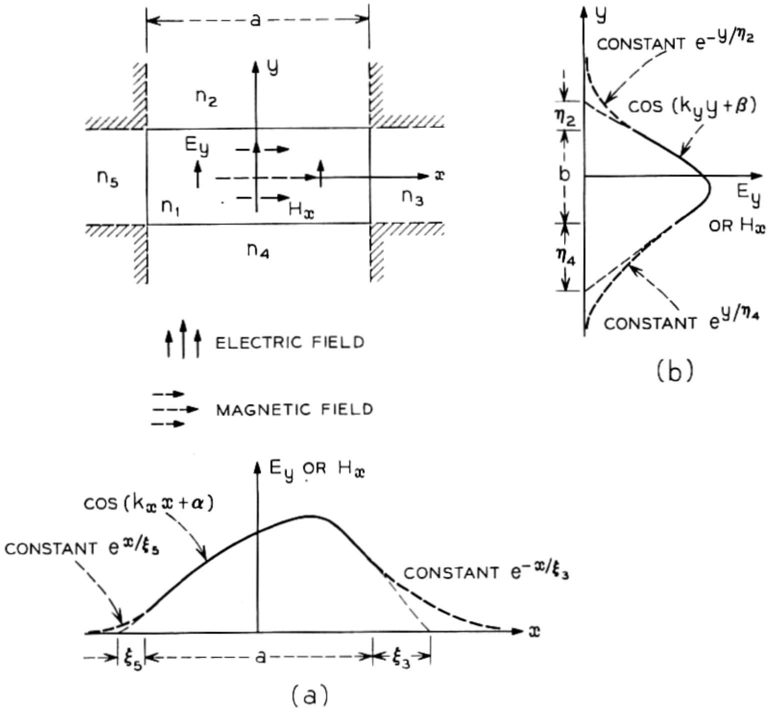


Fig. 4 — Guide immersed in different dielectrics: (a) cross section and (b) field distribution of the fundamental mode E_{11}^y.

cavities with rectangular flat mirrors, but our nomenclature is different.[10] The subindexes p and q indicate the number of extrema each component has within the guide.

Now we describe these modes quantitatively by reproducing the propagation constants found for each medium in Section A.1 of the appendix. Let us call k_z the axial propagation constant and $k_{x\nu}$ and $k_{y\nu}$ the transverse propagation constants along the x and the y directions, respectively, in the νth medium ($\nu = 1, 2, \cdots 5$). Furthermore, let us call

$$k_\nu = kn_\nu = \frac{2\pi}{\lambda} n_\nu \tag{2}$$

the propagation constant of a plane wave in a medium of refractive index n_ν and free-space wavelength λ.

According to equations (39) through (52)

$$k_z = (k_1^2 - k_x^2 - k_y^2)^{\frac{1}{2}} \tag{3}$$

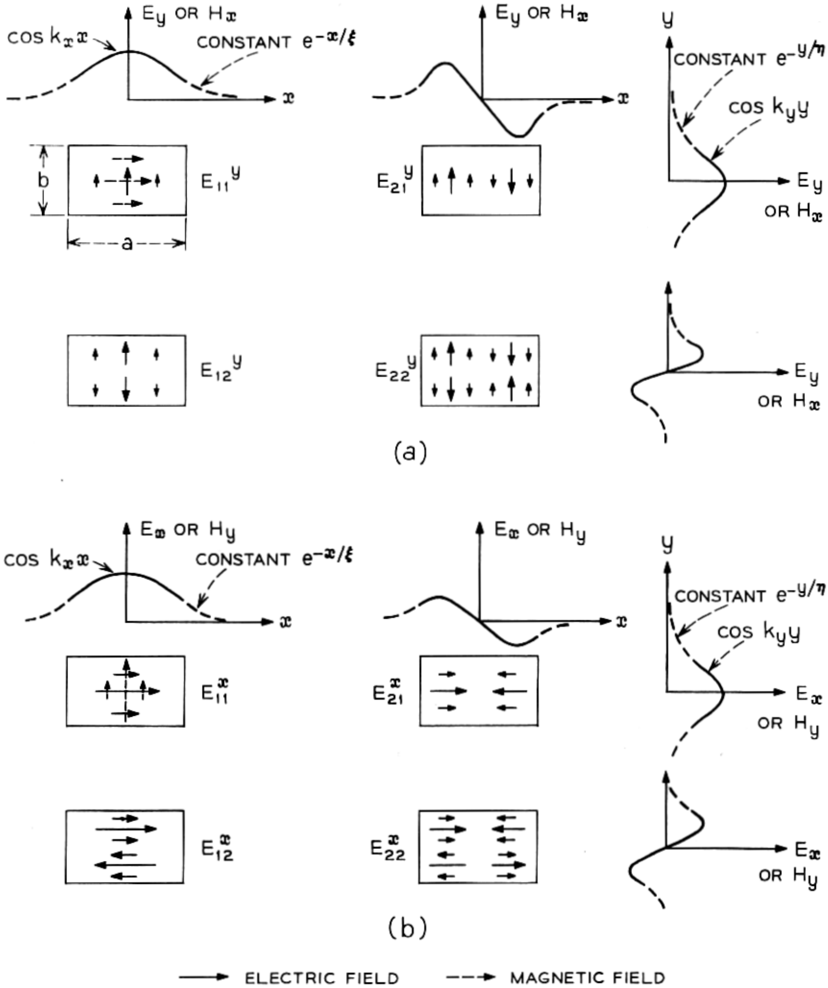


Fig. 5 — (a) Field configuration of E_{pq}^y modes. (b) Field configuration of E_{pq}^x modes.

in which

$$k_x = k_{x1} = k_{x2} = k_{x4} \tag{4}$$

and

$$k_y = k_{y1} = k_{y3} = k_{y5} \,. \tag{5}$$

This means that the fields in media 1, 2, and 4 have the same x

dependence and similarly those in media 1, 3, and 5 have identical y dependence. These transverse propagation constants are solutions of the transcendental equations:

$$k_x a = p\pi - \tan^{-1} k_x \xi_3 - \tan^{-1} k_x \xi_5 \tag{6}$$

$$k_y b = q\pi - \tan^{-1} \frac{n_2^2}{n_1^2} k_y \eta_2 - \tan^{-1} \frac{n_4^2}{n_1^2} k_y \eta_4 \tag{7}$$

in which

$$\xi_{\substack{3\\5}} = \frac{1}{\left| k_{x\substack{3\\5}} \right|} = \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{3\\5}}}\right)^2 - k_x^2\right]^{\frac{1}{2}}} \tag{8}$$

$$\eta_{\substack{2\\4}} = \frac{1}{\left| k_{y\substack{2\\4}} \right|} \quad \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{2\\4}}}\right)^2 - k_y^2\right]^{\frac{1}{2}}} \tag{9}$$

and

$$A_{2,3,4,5} = \frac{\pi}{(k_1^2 - k_{2,3,4,5}^2)^{\frac{1}{2}}} = \frac{\lambda}{2(n_1^2 - n_{2,3,4,5}^2)^{\frac{1}{2}}}. \tag{10}$$

In the transcendental equations (6) and (7), a and b are the transverse dimensions of the guiding rod, and the $\tan^{-1}$ functions are to be taken in the first quadrant.

What are the physical meanings of $\xi_{\substack{3\\5}}$, $\eta_{\substack{2\\4}}$, and $A_{2,3,4,5}$? The amplitude of each field component in medium 3 (Fig. 4) decreases exponentially along x. It decays by $1/e$ in a distance $\xi_3 = 1/\left| k_{x3} \right|$. Similarly ξ_5, η_2, and η_4 measure the "penetration depths" of the field components in media 5, 2, and 4, respectively.

The meaning of A_2 is the following. Consider a symmetric slab derived from Fig. 4 by choosing $a = \infty$ and $n_2 = n_4$. The maximum thickness for which the slab supports only the fundamental mode is A_2.

Expressions (3), (8), and (9) contain k_x and k_y, which are solutions of the transcendental equations (6) and (7). These cannot be solved exactly in closed form. Nevertheless, for well-guided modes, most of the power travels within medium 1, implying

$$\left(\frac{k_x A_{\substack{3\\5}}}{\pi}\right)^2 \ll 1 \quad \text{and} \quad \left(\frac{k_y A_{\substack{2\\4}}}{\pi}\right)^2 \ll 1. \tag{11}$$

It is possible then to solve those transcendental equations in closed, though approximate, form. Their solutions are

$$k_x = \frac{p\pi}{a}\left(1 + \frac{A_3 + A_5}{\pi a}\right)^{-1} \tag{12}$$

$$k_y = \frac{q\pi}{b}\left(1 + \frac{n_2^2 A_2 + n_4^2 A_4}{\pi n_1^2 b}\right)^{-1}. \tag{13}$$

For large a and b, the electrical width, $k_x a$, and the electrical height, $k_y b$, of the guide are close to $p\pi$ and $q\pi$, respectively.

Substituting equations (12) and (13) in equations (3), (8), and (9), we obtain explicit expressions for k_z, ξ_3, ξ_5, η_2, and η_4:

$$k_z = \left[k_1^2 - \left(\frac{\pi p}{a}\right)^2\left(1 + \frac{A_3 + A_5}{\pi a}\right)^{-2} - \left(\frac{\pi q}{b}\right)^2\left(1 + \frac{n_2^2 A_2 + n_4^2 A_4}{\pi n_1^2 b}\right)^{-2}\right]^{\frac{1}{2}} \tag{14}$$

$$\xi_{\substack{3\\5}} = \frac{A_{\substack{3\\5}}}{\pi}\left[1 - \left(\frac{pA_{\substack{3\\5}}}{a} \frac{1}{1 + \dfrac{A_3 + A_5}{\pi a}}\right)^2\right]^{-\frac{1}{2}} \tag{15}$$

$$\eta_{\substack{2\\4}} = \frac{A_{\substack{2\\4}}}{\pi}\left[1 - \left(\frac{qA_{\substack{2\\4}}}{b} \frac{1}{1 + \dfrac{n_2^2 A_2 + n_4^2 A_4}{\pi n_1^2 b}}\right)^2\right]^{-\frac{1}{2}}. \tag{16}$$

3.2 *The* E^x_{pq} *Modes*

Except for the fact that the main transverse components are E_x and H_y, the E^x_{pq} modes are qualitatively similar to the E^y_{pq} modes (Fig. 5b); they differ quantitatively. Distinguishing with bold-face type the symbols corresponding to E^x_{pq} modes, the axial propagation constant and the "penetration depth" in media 2, 3, 4, and 5 are, according to equations (60), (63), and (64),

$$\mathbf{k}_z = (k_1^2 - \mathbf{k}_x^2 - \mathbf{k}_y^2)^{\frac{1}{2}} \tag{17}$$

$$\boldsymbol{\xi}_{\substack{3\\5}} = \frac{1}{|\mathbf{k}_{x\substack{3\\5}}|} = \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{3\\5}}}\right)^2 - \mathbf{k}_x^2\right]^{\frac{1}{2}}} \tag{18}$$

$$\boldsymbol{\eta}_{\substack{2\\4}} = \frac{1}{|\mathbf{k}_{y\substack{2\\4}}|} = \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{2\\4}}}\right)^2 - \mathbf{k}_y^2\right]^{\frac{1}{2}}} \tag{19}$$

in which $\mathbf{k}_x$ and $\mathbf{k}_y$ are solutions of the transcendental equations

$$\mathbf{k}_x a = p\pi - \tan^{-1}\frac{n_3^2}{n_1^2}\mathbf{k}_x\xi_3 - \tan^{-1}\frac{n_5^2}{n_1^2}\mathbf{k}_x\xi_5 \tag{20}$$

$$\mathbf{k}_y b = q\pi - \tan^{-1}\mathbf{k}_y\mathbf{n}_2 - \tan^{-1}\mathbf{k}_y\mathbf{n}_4 . \tag{21}$$

The approximate closed form solutions of these equations are

$$\mathbf{k}_x = \frac{p\pi}{a}\left(1 + \frac{n_3^2A_3 + n_5^2A_5}{\pi n_1^2 a}\right)^{-1} \tag{22}$$

and

$$\mathbf{k}_y = \frac{q\pi}{b}\left(1 + \frac{A_2 + A_4}{\pi b}\right)^{-1}. \tag{23}$$

Substituting these expressions in equations (17), (18), and (19), we derive the explicit results:

$$\mathbf{k}_z = \left[k_1^2 - \left(\frac{\pi p}{a}\right)^2\left(1 + \frac{n_3^2A_3 + n_5^2A_5}{\pi n_1^2 a}\right)^{-2} - \left(\frac{\pi q}{b}\right)^2\left(1 + \frac{A_2 + A_4}{\pi b}\right)^{-2}\right]^{\frac{1}{2}} \tag{24}$$

$$\xi_{\substack{3\\5}} = \frac{A_{\substack{3\\5}}}{\pi}\left[1 - \left(\frac{pA_{\substack{3\\5}}}{a}\,\frac{1}{1 + \dfrac{n_3^2A_3 + n_5^2A_5}{\pi n_1^2 a}}\right)^2\right]^{-\frac{1}{2}} \tag{25}$$

$$\mathbf{n}_{\substack{2\\4}} = \frac{A_{\substack{2\\4}}}{\pi}\left[1 - \left(\frac{qA_{\substack{2\\4}}}{b}\,\frac{1}{1 + \dfrac{A_2 + A_4}{\pi b}}\right)^2\right]^{-\frac{1}{2}}. \tag{26}$$

If

$$\frac{1}{n_1}\left(n_1 - n_{\substack{2\\3\\4\\5}}\right) \ll 1,$$

these results coincide with those in equations (14), (15), and (16), indicating that the E_{pq}^x and E_{pq}^y modes become degenerate.

3.3 *Examples*

The axial propagation constants k_z and $\mathbf{k}_z$, given in equations (3) and (17) and properly normalized, have been plotted in Figs. 6a through k as a function of the normalized height of the guide

$$\frac{b}{A_4} = \frac{2b}{\lambda}(n_1^2 - n_4^2)^{\frac{1}{2}}$$

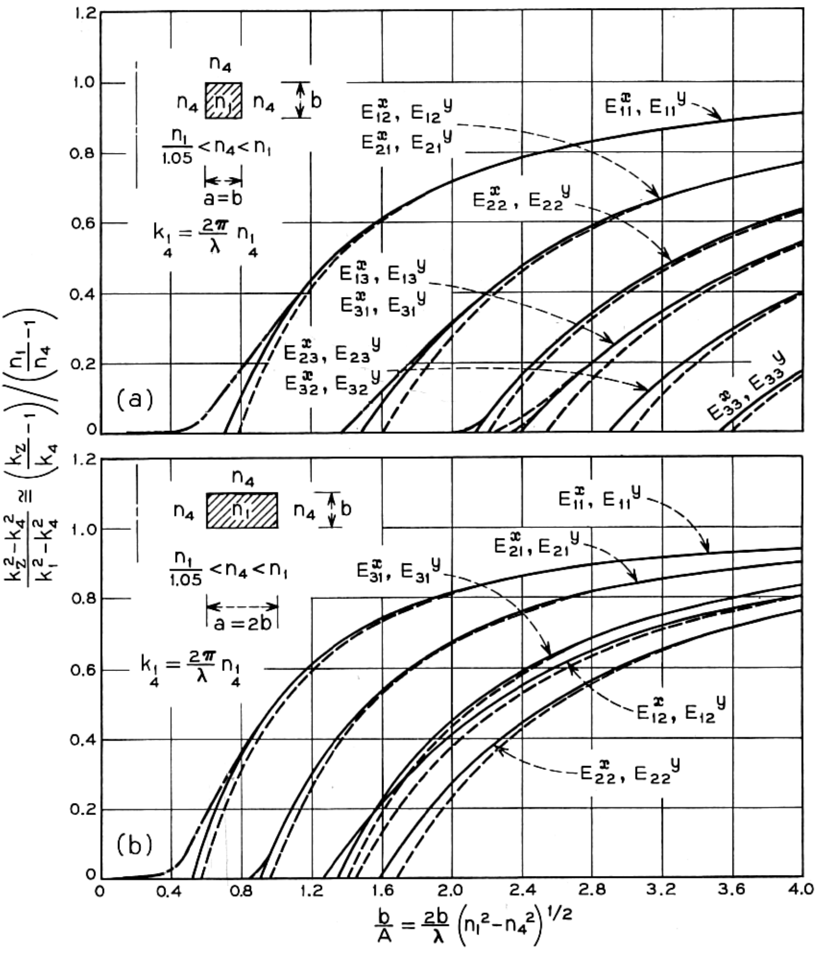


Fig. 6 — Propagation constant for different modes and guides. ———— transcendental equation solutions; — — — — closed form solutions; —·—·— Goell's computer solutions of the boundary value problem.

for several geometries and surrounding media.* The ordinate in each of these figures is

$$\frac{k_z^2 - k_4^2}{k_1^2 - k_4^2};$$

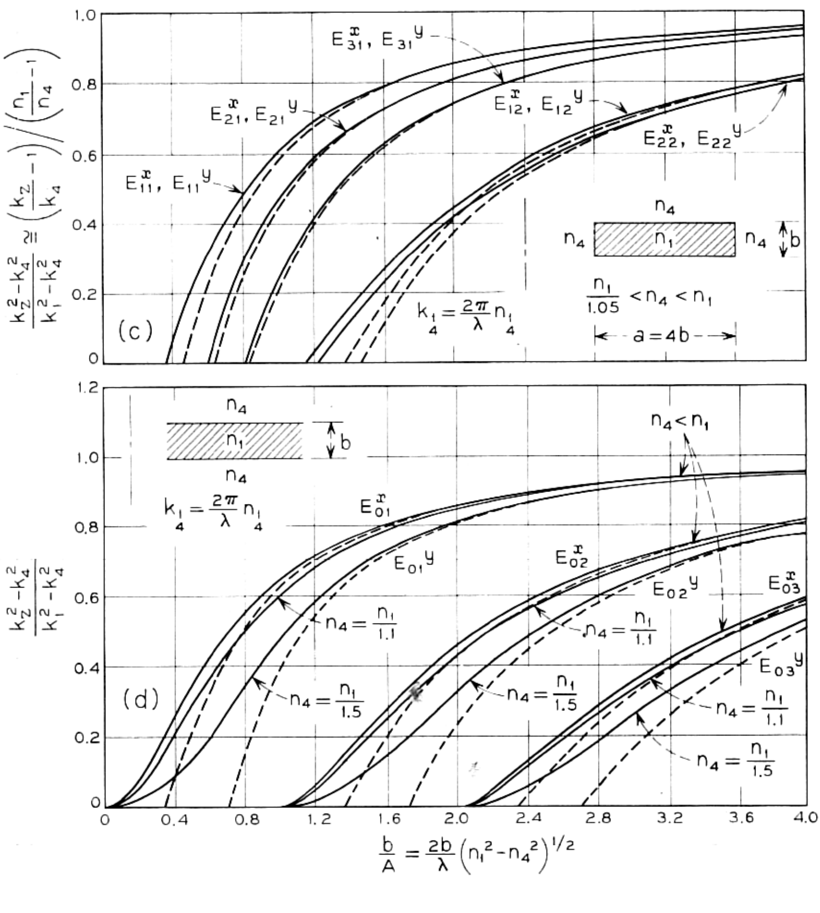
it varies between 0 and 1. It is 0 when $k_z = k_4$, that is, when the guide

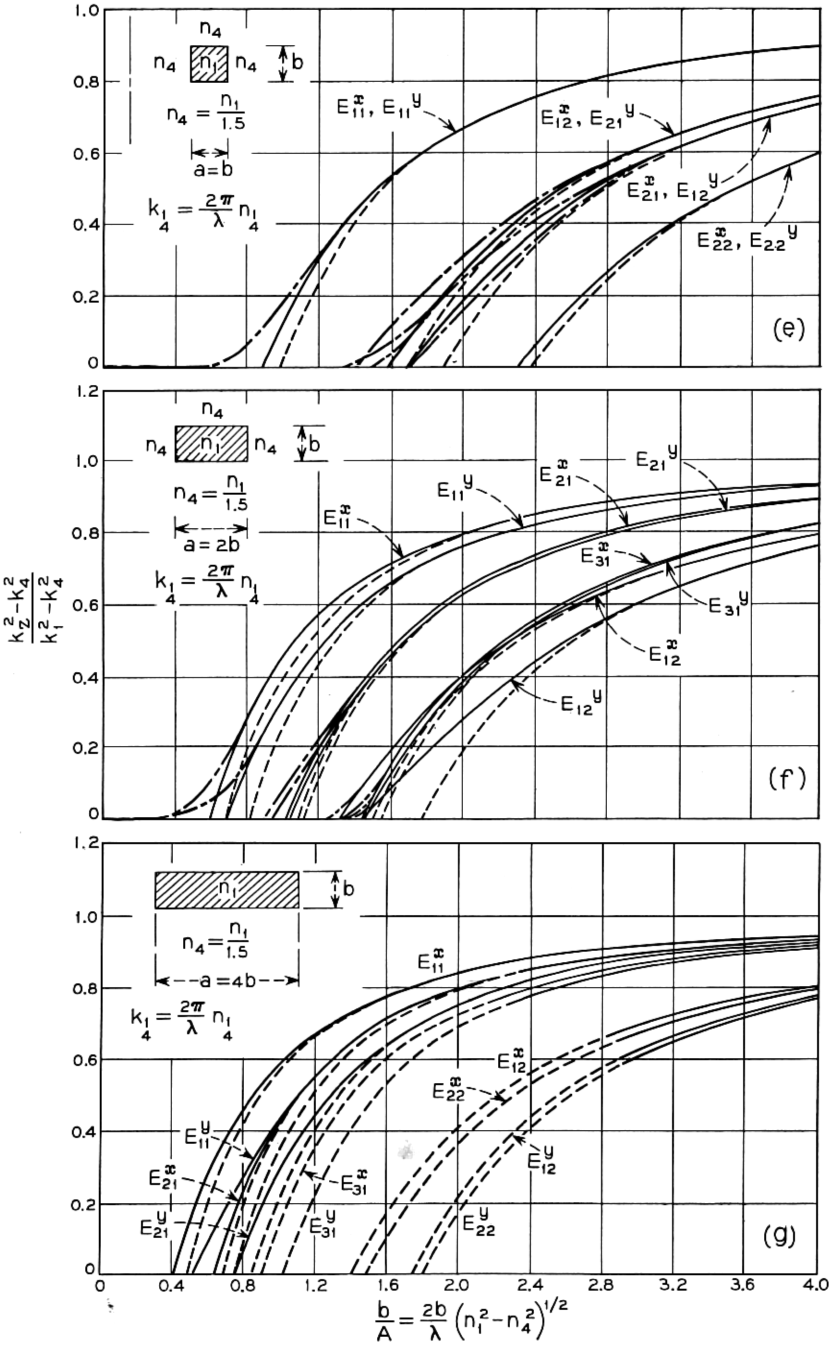
* In these figures we use the same symbol k_z for both the E_{pq}^y and the E_{pq}^x modes.

is so small that the mode under consideration becomes unguided or, in other words, the "penetration depth" in medium 4 is ∞. It is 1 when the guide is so large that $k_z = k_1$, which means that all the field travels within the guiding rod and the "penetration depths" in media 2, 3, 4, and 5 are zero.

The solid curves have been obtained using the exact numerical solutions of the transcendental equations (6), (7), (20), and (21); for the transverse propagation constants k_x and k_y; the dashed lines have been derived using the closed form approximations (12), (13), (22), and (23). In Figs. 6a, 6b, 6e, and 6f, for comparison, we have also included the dotted-dashed lines which are the results obtained by Goell as computer solutions of the boundary value problem.[4]

The three solutions coincide even for moderately large values of b.




n4
n4 n1 n4 b
n4 = n1/1.5
a=b
k1/4 = 2π/λ n1/4
E11x, E11y
E12x, E21y
E21x, E12y
E22x, E22y
(e)
a = 2b
E11x
E11y
E21x
E21y
E31x
E31y
E12x
E12y
(f)
a = 4b
E22x
E31x
E31y
E22y
(g)
(kz² − k4²)/(k1² − k4²)
b/A = 2b/λ (n1² − n4²)^1/2
0 0.2 0.4 0.6 0.8 1.0 1.2
0 0.4 0.8 1.2 1.6 2.0 2.4 2.8 3.2 3.6 4.0

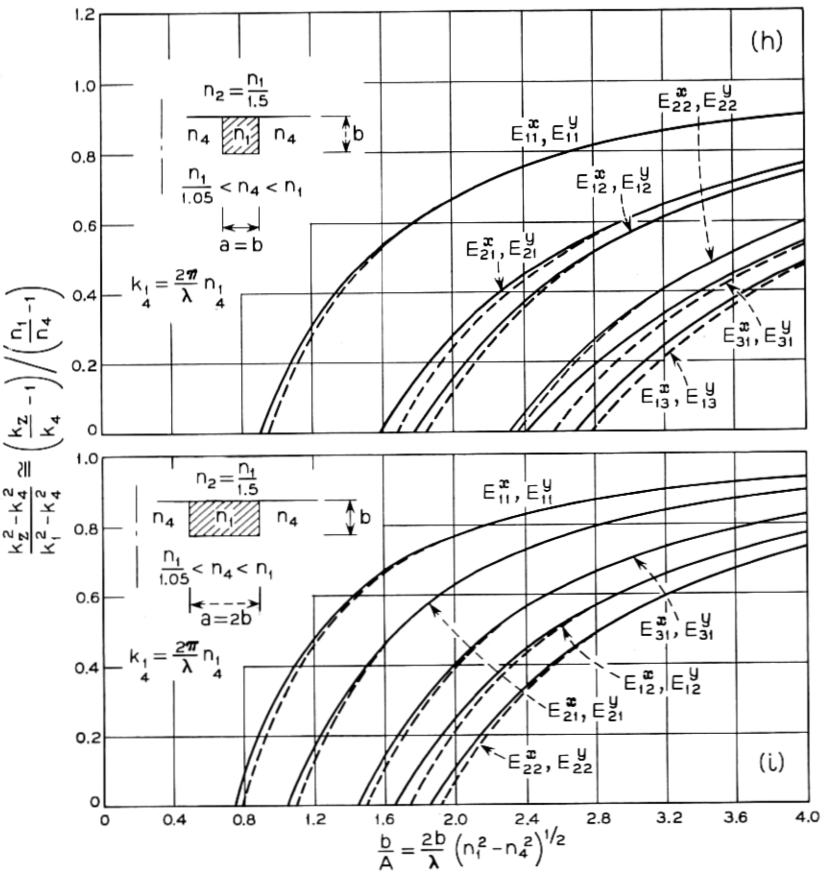
Thus, for a guide and mode for which

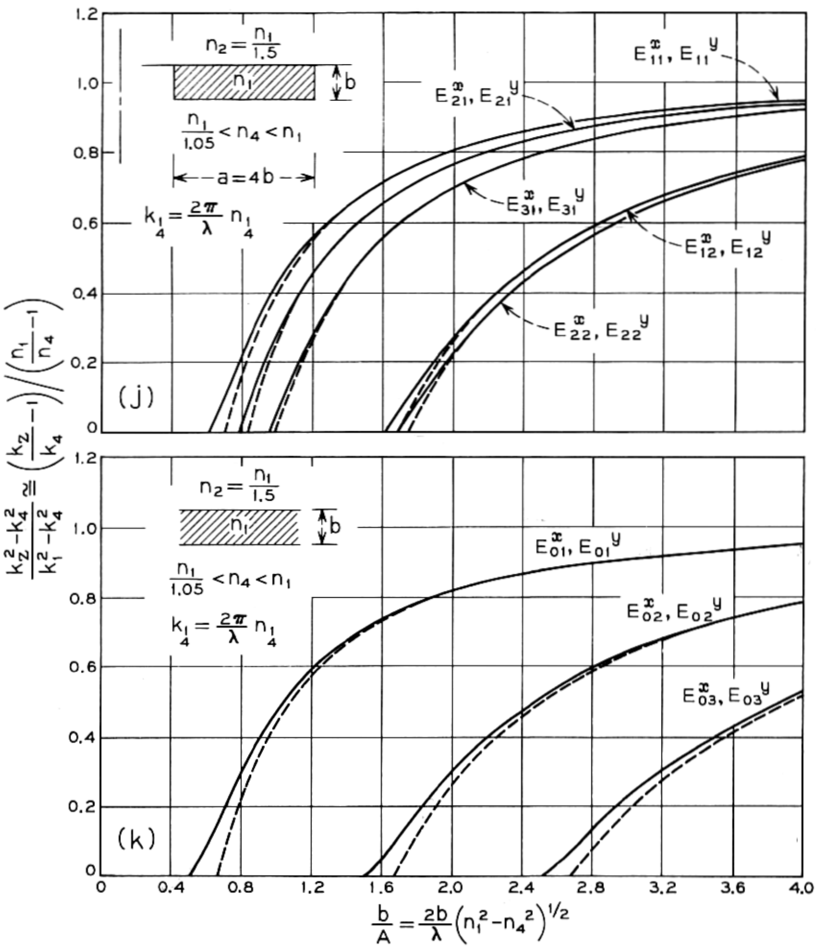
$$\frac{k_z^2 - k_4^2}{k_1^2 - k_4^2} \geqq 0.5,$$

the closed form approximation is within a few percent of the exact value. This gives us confidence to use our results in guides with an aspect ratio $a/b > 2$, in guides surrounded by several dielectrics and in directional couplers for which there are no computer calculations available.

The largest discrepancy between our results and Goell's occurs for

$$\frac{k_z^2 - k_4^2}{k_1^2 - k_4^2} \cong 0$$





and especially for the fundamental modes E^x_{11} and E^y_{11} . Our approximate theory is incapable of predicting the fact that these modes remain guided no matter how small the guide's cross section.

Figures 6a through d cover the cases of rectangular guides totally embedded in a single dielectric of slightly lower refractive index. For all practical purposes, given p and q, the E^x_{pq} and E^y_{pq} modes are degenerate, and the square cross section provides the widest separation between modes.

Figures 6e through g also consider rectangular guides embedded in a single dielectric, but the external refractive index is 1.5 times smaller than the internal one. A glass rod immersed in air is an example. The substantial difference of refractive indexes breaks the degeneracy for any rectangular cross section. Rectangular waveguides as in Fig. 1a, with three sides in contact with slightly lower refractive indexes and the fourth side in contact with air, are covered in Fig. 6h through k.

The approximate dispersion relation (14) for E^y_{pq} modes, in a rectangular guide surrounded by four different dielectrics, has been put in graphical form in Fig. 7 by plotting the equivalent equation

$$p^2 X + q^2 Y = 1 \tag{27}$$

in which

$$X = \left(\frac{\pi}{a}\right)^2 \left(1 + \frac{A_3 + A_5}{\pi a}\right)^{-2} (k_1^2 - k_z^2)^{-1} \tag{28}$$

and

$$Y = \left(\frac{\pi}{b}\right)^2 \left(1 + \frac{n_2^2 A_2 + n_4^2 A_4}{\pi n_1^2 b}\right)^{-2} (k_1^2 - k_z^2)^{-1}. \tag{29}$$

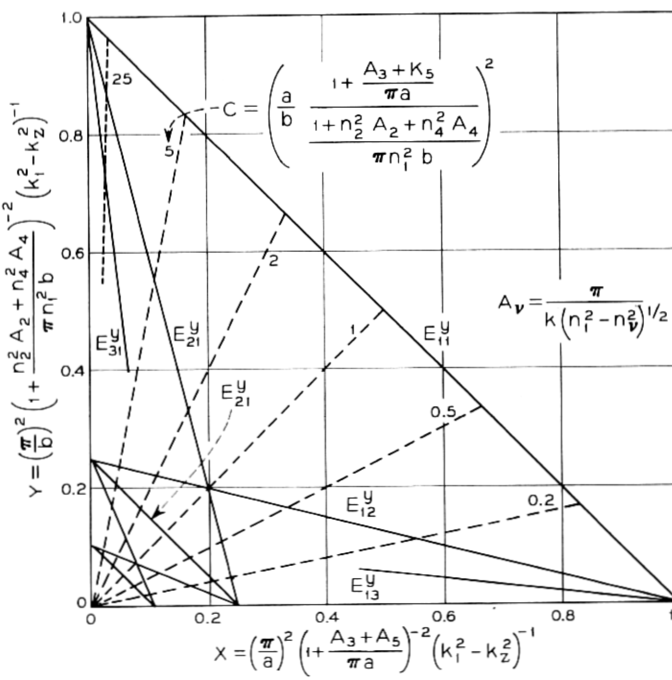


Fig. 7 — Nomograph to dimension a guide immersed in several dielectrics in such a way that it supports any prescribed number of modes.

The curves plotted for different values of p and q are straight lines (solid lines); since the values of X and Y are physically meaningful when they are positive, the plots are kept within the first quadrant.

In Fig. 7 the dotted lines depict the equation

$$\frac{Y}{X} = \left[\frac{a}{b} \frac{1 + \dfrac{A_3 + A_5}{\pi a}}{1 + \dfrac{n_2^2 A_2 + n_4^2 A_4}{\pi n_1^2 b}} \right]^2 = C. \tag{30}$$

Given any guide, we can calculate C which is a function of the dimensions, refractive indexes, and wavelength. The corresponding dotted line intersects all the solid lines representing the different modes. The abscissa or ordinate of each intersection yields, after some algebra, the propagation constant k_z of each particular mode. If the resulting k_z is smaller than the smallest k_ν, that mode is not guided.

Another way of using the graph is this: Suppose one wants a guide with such dimensions that at a given wavelength only the E_{11}^y mode is supported. Picking $k_z = k_{\nu\min}$, any combination of n_1, n_2, n_3, n_4, n_5, a, and b represented by a point within the triangle limited by the solid lines E_{11}^y, E_{12}^y, and E_{21}^y will satisfy the proposed single-mode requirement.

In the graph it is enough to substitute a by b and everything we said about E_{pq}^y modes is applicable to E_{pq}^x modes.

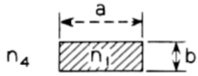
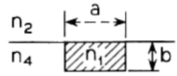
Figures 6a through k have been used to determine dimensions for several guides. All of them have the maximum dimensions compatible with exclusive guidance of the E_{11}^x and E_{11}^y modes. The results are collected in Table I.

In general, the geometry with $n_2 < n_4$ requires a larger waveguide cross section than with $n_2 = n_4$. This means reducing the refractive index on one side of the guide reduces its ability to guide. The explanation of this paradox is found in the known fact that a symmetric slab indeed guides "better" than an asymmetric one. Comparing, for example, Figs. 6d and 6k, in which the solid curves have been drawn solving Maxwell's equations exactly, the E_{p1}^x and E_{p1}^y modes can be guided by the symmetric slab (Fig. 6d) no matter how small the thickness b; there is a minimum thickness required for the asymmetric slab (Fig. 6k) to guide the same modes.[9]

Consider the guide immersed in a single dielectric. In general, the guide's height b is inversely proportional to

$$\frac{1}{(n_1^2 - n_4^2)^{\frac{1}{2}}}.$$

TABLE I—TYPICAL DIMENSIONS FOR SEVERAL GUIDES*

							
	$\frac{n_1}{n_4} = 1.001$	$\frac{n_1}{n_4} = 1.01$	$\frac{n_1}{n_4} = 1.05$	$\frac{n_1}{n_4} = 1.5$	$\frac{n_1}{n_2} = 1.5$; $\frac{n_1}{n_4} = 1.001$	$\frac{n_1}{n_2} = 1.5$; $\frac{n_1}{n_4} = 1.01$	$\frac{n_1}{n_2} = 1.5$; $\frac{n_1}{n_4} = 1.05$
a = b	15.3†	4.9	2.25	0.92	17.7	5.6	2.6
a = 2b	19	6.1	2.8	1.21	23.2	7.4	3.4
a = 4b	26.8	8.5	3.8	1.37	34.9	11	4.9

* Dimensions are for guides capable of supporting only the fundamental modes E_{11}^x and E_{11}^y.
† All numbers in the table must be multiplied by λ/n_1.

For $n_1 = 1.5$, $n_4 = 1$, and $\lambda = 1\mu$, the largest guide height corresponds to the square cross section, and $b = a = 0.61\mu$. This dimension may be too small and difficult to control. The tolerance requirements may be relaxed by choosing $n_1 - n_4 \ll 1$. Nevertheless, this difference cannot be made arbitrarily small because the guide loses its ability to negotiate sharp bends.[11]

In all these examples the fundamental modes E_{11}^x and E_{11}^y are almost degenerate, so symmetry imperfections of the guide tend to couple these modes. A lossy layer, added to one of the interfaces between guiding rod and surrounding dielectrics, should attenuate the mode with polarization parallel to that interface. As an alternative, the guide can be made to support only the fundamental mode E_{11}^y by substituting medium 2 with a low impedance medium such as a dielectric with large refractive index or a metal.

An example of such a guide and the propagation constant of its modes are shown in Fig. 8. By choosing

$$a < \frac{0.7\lambda}{(n_1^2 - n_4^2)^{\frac{1}{2}}}$$

only the E_{11}^y mode is guided. If the metal is not perfect, there is power leakage into the low impedance medium. The smaller that impedance, the smaller the leakage.

Guides for integrated optics may be easier to build with $a/b \gg 1$. We can use Fig. 7 to design a guide of arbitrary dimensions a and b which is

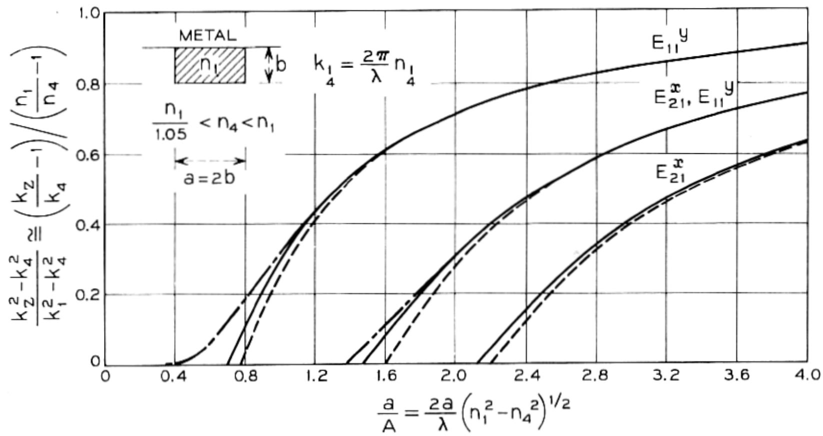


Fig. 8 — Propagation constant for modes in a guide surrounded by metal and dielectrics. ———— transcendental equation solutions; – – – – closed form solutions; —·—·— Goell's computer solutions of the boundary value problem.

still capable of supporting only the E^x_{11} and E^y_{11} modes. An as example, let us calculate what the values

$$n_3 = n_5 = n_1(1 + \Delta) \quad \text{and} \quad n_2 = n_4 = n_1(1 + \Delta')$$

should be, assuming

$$\Delta,\ \Delta' \ll 1, \quad \text{and} \quad \frac{a}{b} = 5.$$

Choosing

$$\left(\frac{\Delta'}{\Delta}\right)^{\frac{1}{2}} = \frac{a}{b} = 5, \tag{31}$$

one derives from Fig. 7

$$C = \left(\frac{a}{b}\right)^2 = 25.$$

The curve corresponding to $C = 25$ has been plotted as a dotted line in Fig. 7. It intercepts the E^y_{21} line at

$$Y = \left[\frac{b}{\pi} + \frac{1}{\pi k n_1}\left(\frac{2}{\Delta'}\right)^{\frac{1}{2}}\right]^{-2}(k_1^2 - k_z^2)^{-1} = 0.88.$$

In this expression, by making

$$k_z = kn_1(1 - \Delta),$$

the guide supports only the E^y_{11} and E^x_{11} modes; its height is then

$$b = 1.66 \frac{\lambda}{n_1(\Delta')^{\frac{1}{2}}}\cdot \tag{32}$$

We can choose b arbitrarily by the proper selection of $\triangle'$.

For

$$\lambda = 1\mu\ n_1 = 1.5, \text{ and } b = 5\mu,$$

from equations (31) and (32) we obtain

$$a = 25\mu,\ \Delta = 0.002, \text{ and } \Delta' = 0.05.$$

IV. DIRECTIONAL COUPLER

In general, the directional coupler can transmit E^x_{pq} and E^y_{pq} modes; but if the sides a and b of the guides are selected small enough, only the fundamental modes E^x_{11} and E^y_{11} are guided. Let us concentrate on the E^y_{11} mode. The coupler guides two kinds of E^y_{11} modes: one is symmetric (Fig. 9c) while the other is antisymmetric (Fig. 9d). Both are essentially TEM modes with main field components E_y and H_x. The electric and magnetic field intensity profiles for both modes are depicted qualitatively in Figs. 9b, c, and d.

Ignoring the small effects introduced by the loose coupling, the electrical width $k_x a$ and height $k_y b$ of each guide, as well as the field penetrations ξ_3 and η_2, coincide with those of the guide described in Section III. Similar reasoning applies to the E^x_{11} mode.

The coupling coefficient K between the two guides and the length L necessary for complete transfer of power from one to the other are, according to equations (56) and (59),[12]

$$-iK = \frac{\pi}{2L} = 2\frac{k_x^2}{k_z}\frac{\xi_5}{a}\frac{\exp(-c/\xi_5)}{1 + k_x^2\xi_5^2}. \tag{33}$$

For E^y_{pq} modes, k_z and ξ_5 are given in equations (3) and (8), and k_x is the solution of equation (6). Similarly, for E^x_{pq} modes, k_z, ξ_5, and k_x are obtained from equations (17), (18), and (20). As expected, the coupling decreases exponentially with the ratio c/ξ_5 between the guide's separation and the field penetration in medium 5.

The normalized coupling coefficient

$$\frac{|K|\,a}{\left[1 - \left(\frac{n_5}{n_1}\right)^2\right]^{\frac{1}{2}}}\frac{k_z}{k_1} = \frac{\pi}{2}\frac{a}{L}\frac{1}{\left[1 - \left(\frac{n_5}{n_1}\right)^2\right]^{\frac{1}{2}}}\frac{k_z}{k_1}$$

$$= 2\left(\frac{k_x A_5}{\pi}\right)^2\left[1 - \left(\frac{k_x A_5}{\pi}\right)^2\right]^{\frac{1}{2}} \exp\left\{-\pi\frac{c}{A_5}\left[1 - \left(\frac{k_x A_5}{\pi}\right)^2\right]^{\frac{1}{2}}\right\} \tag{34}$$

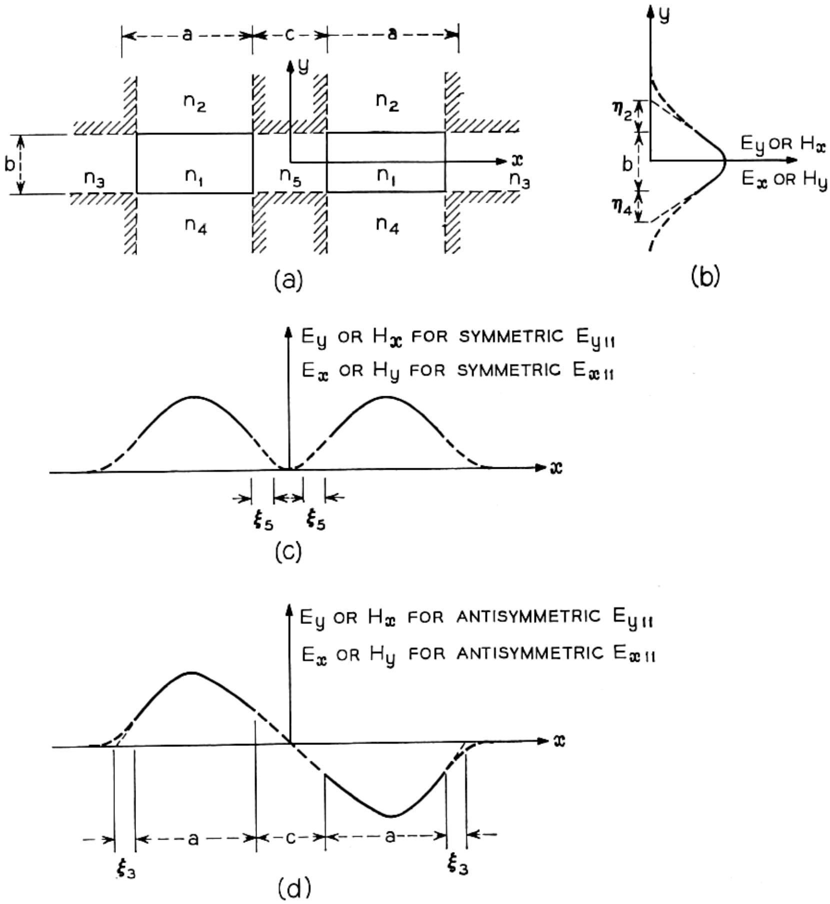


Fig. 9—Directional coupler immersed in several dielectrics: (a) cross section, (b), (c), and (d) field distributions.

derived from equation (33) by substituting ξ_5 for its value given in equation (8) has been plotted in Fig. 10 for the E^y_{1q} mode, assuming $n_3 = n_5$ and n_1/n_5 is arbitrary. The solid and dotted lines were obtained using the exact solution of (6) and the approximate expression (12), respectively, for k_x. Both sets of curves are close to each other, especially for $2a/\lambda(n_1^2 - n_5^2)^{\frac{1}{2}} \geqq 1$.

The dashed-dotted lines are the couplings obtained by A. L. Jones[5] for two parallel cylinders of refractive index $n_1 = 1.8$ embedded in a medium $n_5 = 1.5$.[5] As expected, if the diameters of the round guides are equal to the widths of the rectangular guides, and if the separations are the same, the coupling between the round guides should be slightly smaller than that between the rectangular ones.

The normalized coupling equation (34) for the E^x_{1q} mode has been plotted in Fig. 11, using for k_x the exact solution of equation (20). For n_1/n_5 close to unity, the lines get close to the solid curves in Fig. 10 as the E^y_{1q} and E^x_{1q} modes approach degeneracy. The influence of the height b of the guides, the refractive indices n_2 and n_4, and the value of q in the coupling of either mode is not important since they only affect k_z.

To work some examples, assume

$$n_1 = 1.5, \qquad n_2 = n_3 = n_4 = n_5 = \frac{1.5}{1.01}, \quad \text{and} \quad a = 2b.$$

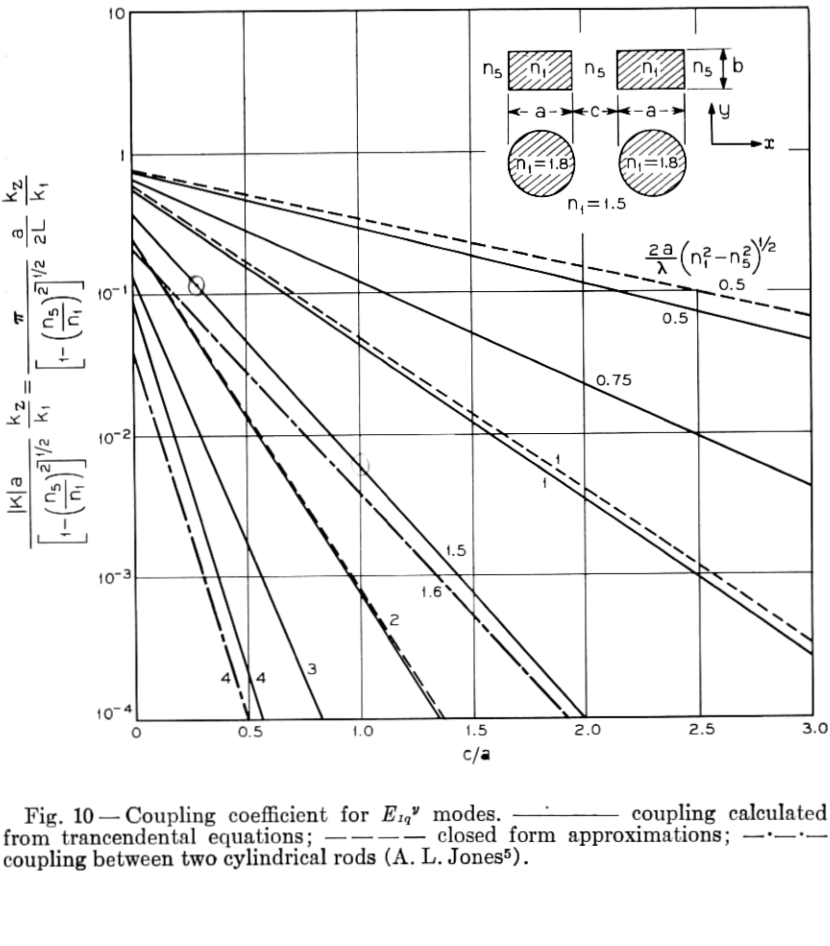


Fig. 10 — Coupling coefficient for $E_{1q}{}^y$ modes. ——— coupling calculated from trancendental equations; — — — — closed form approximations; —·—·— coupling between two cylindrical rods (A. L. Jones[5]).

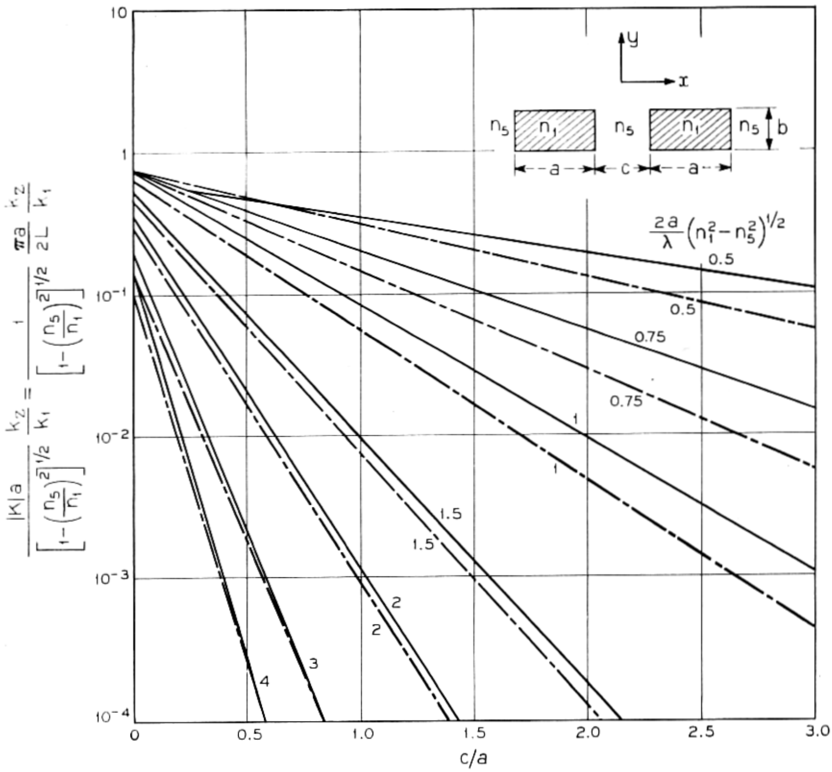


Fig. 11 — Coupling coefficient for E_{1q}^x modes. ———— E_{1q}^x coupling for $n_1/n_5 = 1.5$; —·—·— E_{1q}^x coupling for $n_1/n_5 = 1.1$.

To insure that each guide only supports the E_{11}^x and E_{11}^y modes, the normalized dimension b according to Fig. 6b must be chosen to be

$$\frac{2b}{\lambda}(n_1^2 - n_4^2)^{\frac{1}{2}} = 0.75.$$

Consequently

$$b = 1.77\lambda, \qquad a = 3.54\lambda, \quad \text{and} \quad \frac{k_z}{k_1} \cong 1.$$

From Fig. 10 we obtain the coupler length L for complete power transfer:

$$L = 6540\lambda \quad \text{for} \quad c = a \quad \text{and} \quad L = 262\lambda \quad \text{for} \quad c = \frac{a}{4}.$$

How far apart should two guides of length l be spaced to have small coupling? If the transfer coefficient $|T| = l|K| \ll 1$, from equation (33) we derive

$$c = \xi_5 \log \left[2 \frac{l}{|T|} \frac{k_x^2}{k_z} \frac{\xi_5}{a} \frac{1}{1 + k_x^2 \xi_5^2} \right]. \quad (35)$$

For the same guide dimensions of the previous example and for

$$l = 1 \text{ cm}, \lambda = 1\mu, \quad \text{and} \quad T = 0.01,$$

we derive, from either equation (35) or Fig. 10, that c/a = 2.5. Consequently, both guides 3.54μ wide and 1 cm long would couple −40dB if their separtion is 8.9μ.

Now we evaluate how a small change of the refractive index between the guides modifies their coupling. Such would be the case if the medium between the guides is, for example, an electrooptic material and we change the applied field to modulate or switch the output.

For E_{11}^x and E_{11}^y modes, assuming well-guided modes ($k_x A_5/\pi \ll 1$) and $n_1 - n_5/n_1 \ll 1$, the ratio between couplings for two values of refractive index in medium 5 (for example, n_5 and $n_5(1 + \delta)$), result from equations (34) and (12):

$$\frac{K_1}{K_2} = \frac{L_2}{L_1} = \exp\left\{ -\pi \left(\frac{n_1^2}{n_5^2} - 1 \right)^{-1} \frac{c\delta}{A_5} \left[1 - \left(\frac{2}{\pi} + \frac{a}{A_5} \right)^{-2} \right]^{\frac{1}{2}} \right\}. \quad (36)$$

That ratio is 1/2 if

$$\delta = 0.22 \left(\frac{n_1^2}{n_5^2} - 1 \right) \frac{A_5}{c} \left[1 - \left(\frac{2}{\pi} + \frac{a}{A_5} \right)^{-2} \right]^{-\frac{1}{2}}. \quad (37)$$

A directional coupler with coupling coefficient K_1 and length $L = \pi/|2K_1|$ would transfer all the power from one guide to the other. If the refractive index of the medium between the guides was changed from n_5 to $n_5(1+\delta)$ such that equation (37) is satisfied, the power would emerge at the end of the input guide. The larger the separation c of the guides, and the smaller the difference of refractive indexes $n_1 - n_5$, the smaller the change of refractive index required.

Following the example above, for

$$n_1 = 1.5, \qquad n_2 = n_3 = n_4 = n_5 = \frac{1.5}{1.01},$$

$$a = 1.5A_5 = 3.54\lambda, \quad \text{and} \quad c = a,$$

the percentage change of index required is only $\delta = 0.0033$.

V. DIRECTIONAL COUPLER MADE WITH SLIGHTLY DIFFERENT GUIDES

Consider the directional coupler of Fig. 12 in which the two guides have slightly different heights: one measures $b + h$ and the other $b - h$.

Let us qualitatively plot the coupling coefficient as a function of h, Fig. 13. Because of simple arguments of symmetry, the absolute value of coupling coefficient is stationary (first derivative zero) around $h = 0$. Therefore, the coupling coefficient between two guides of height b_1 and b_2 is the same as that of the coupling between two identical guides of height $1/2(b_1 + b_2)$, provided that $|b_1 - b_2|$ is small enough.

This reasoning applies to guides with different widths, heights, and refractive indices, provided that the differences are small enough. Unfortunately, as in most perturbation analysis, we don't know what "small enough" is unless we calculate the next higher order term.

VI. SUMMARY AND CONCLUSIONS

A dielectric rod (Fig. 4a) of rectangular cross section a by b surrounded by different dielectrics supports, through total internal reflection, two families of hybrid modes. They are essentially TEM modes polarized either in the x or the y direction; we call them E^x_{pq} and E^y_{pq}. The subindices state the number of extrema (p in the x direction and q in the y direction) of the magnetic or electric transverse field components.

Dispersion curves for guides of different proportions and different surrounding dielectric are plotted in Figs. 6a through k. Typical dimensions for several guides capable of supporting only the fundamental modes E^x_{11} and E^y_{11} are contained in Table I.

By picking dielectrics with similar indexes, the guide dimensions can be made large compared with λ, thus reducing the tolerance requirements. The dimensions a and b can be picked arbitrarily and still achieve

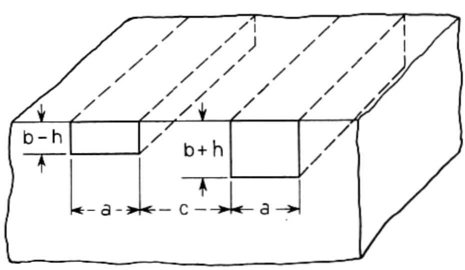


Fig. 12 — Directional coupler with guides of different heights.

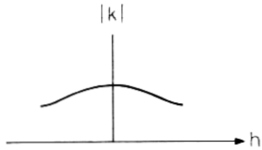


Fig. 13 — Qualitative behavior of the coupling coefficient as a function of h.

a guide which supports only the fundamental modes if one can choose the refractive indexes. The design is achieved with the help of either equation (14) or Fig. 7.

The penalty one pays with most of these guides is that the fundamental modes are almost degenerate; consequently, symmetry imperfections tend to couple them. A lossy layer added to the interface $y = b/2$ (Fig. 4a) should attenuate the E^x_{11} mode more than the E^y_{11}. As an alternative, the guide can be made to support only the E^y_{11} mode by metalizing the same interface. Dispersion curves are shown in Fig. 8.

Since the field is not confined, there is coupling between two of these guides (Fig. 3). Design curves for directional couplers are given in Figs. 10 and 11.

Typically, for $n_1 = 1.5$, $n_2 = n_3 = n_4 = n_5 = 1.5/1.01$, $a = 3.54\lambda$, $b = a/2 = 1.77\lambda$, and $c = a/4 = 0.88\lambda$, according to equation (33) the length necessary for 3dB coupling is $L/2 = 131\lambda$. This length increases exponentially with the separation between the guides.

Increasing the refractive index between the guides by a 3 per thousand doubles the coupling.

What is a reasonable separation to prevent coupling? Using the numbers of the previous example, two parallel guides 1 cm long separated by 2.5 times the width of each guide have a coupling of −40 dB.

The dielectric waveguides and the directional couplers described show great promise as basic elements for integrated optical circuitry because they:

(*i*) Can be made single mode even though their transverse dimensions can be large compared with the free space wavelength of operation. Consequently, the tolerance requirements can be relaxed.

(*ii*) Permit the building of compact optical components.

(*iii*) Are mechanically stable and alignment problems are minimized.

(*iv*) Are relatively simple structures and lend themselves to being fabricated with high precision integrated circuit techniques.

(*v*) Can include active devices of comparable small dimensions.

APPENDIX A

Field Analysis of the Directional Coupler

We solve Maxwell's equations for the directional coupler whose cross section is depicted in Fig. 3. The structure is symmetric with respect to the $x = 0$ plane; therefore, the modes have electric fields which are either symmetric or antisymmetric with respect to that plane. Consequently, the guide we have to study is simpler (Fig 14): if the plane $x = 0$ is an electric short circuit, the modes of the coupler propagating along z are antisymmetric; if the plane $x = 0$ is a magnetic short circuit, the modes are symmetric. As is known, it is the interaction of these symmetric and antisymmetric modes traveling with different phase velocities along z that represents the effect of coupling.

As discussed in Section II, by neglecting the power propagating through the shaded areas, the fields must be matched only along the sides of region 1. We find that two families of modes can satisfy the boundary conditions; we call them E^x_{pq} and E^y_{pq}. Each mode in the first family has most of its electric field polarized in the x direction, while each mode of the second family has the electric field almost completely polarized in the y direction. The subindexes p and q characterize the member of the family by the number of extrema that these transverse field components have along the x and y directions, respectively. For example, the E^x_{11} mode has its electric field virtually along x, its magnetic field along y; the amplitudes of the field have one maximum in each direction.

Each family of modes will be studied separately.

A.1 E^y_{pq} *Modes: Polarization Along y*

The field components in the νth of the five areas in Fig. 14 are:[13]

$$H_{x\nu} = \exp(-ik_z z + i\omega t)\begin{cases} M_1 \cos(k_x x + \alpha)\cos(k_y y + \beta) & \text{for } \nu = 1 \\ M_2 \cos(k_x x + \alpha)\exp(-ik_{y2} y) & \text{for } \nu = 2 \\ M_3 \cos(k_y y + \beta)\exp(-ik_{x3} x) & \text{for } \nu = 3 \\ M_4 \cos(k_x x + \alpha)\exp(ik_{y4} y) & \text{for } \nu = 4 \\ M_5 \cos(k_y y + \beta)\sin(k_{x5} x + \gamma) & \text{for } \nu = 5 \end{cases}$$

$$H_{y\nu} = 0$$

$$H_{z\nu} = -\frac{i}{k_z}\frac{\partial^2 H_{x\nu}}{\partial x\,\partial y} \tag{38}$$

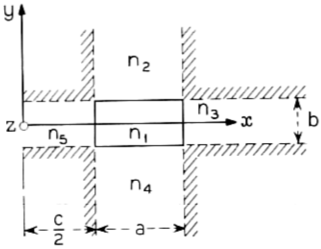


Fig. 14 — Coupler cross section with plane $x = 0$ either an electric or magnetic short circuit.

$$E_{x\nu} = -\frac{1}{\omega\epsilon n_\nu^2 k_z}\frac{\partial^2 H_{x\nu}}{\partial x\,\partial y}$$

$$E_{y\nu} = \frac{k^2 n_\nu^2 - k_{y\nu}^2}{\omega\epsilon n_\nu^2 k_z} H_{x\nu}$$

$$E_{z\nu} = \frac{i}{\omega\epsilon n_\nu^2}\frac{\partial H_{x\nu}}{\partial y}$$

in which M_ν determines the amplitude of the field in the νth medium; α and β locate the field maxima and minima in region 1; γ equal to 0° or 90° implies that the plane $x = 0$ is an electric (antisymmetric mode) or magnetic (symmetric mode) short circuit, respectively; ω is the angular frequency; ϵ and μ (appearing in $k^2 = \omega^2\epsilon\mu$) are the permittivity and permeability of free space.

In the νth medium the refractive index is n_ν, and the propagation constants $k_{x\nu}$, $k_{y\nu}$, and k_z are related by

$$k_{x\nu}^2 + k_{y\nu}^2 + k_z^2 = \omega^2\epsilon\mu n_\nu^2 = k_\nu^2 . \tag{39}$$

To match the fields at the boundaries between the region 1 and the regions 2 and 4, we have assumed in equation (38)

$$k_{x1} = k_{x2} = k_{x4} = k_x \tag{40}$$

and similarly to match the fields between media 1, 3, and 5,

$$k_{y1} = k_{y3} = k_{y5} = k_y . \tag{41}$$

Before finding the characteristic equations, let us assume the refractive index n_1 of the guide to be slightly larger than the others. That is

$$\frac{n_1}{n_{\substack{2\\3\\4\\5}}} - 1 \ll 1. \tag{42}$$

As a consequence only modes made of plane wavelets impinging at grazing angles on the surface of medium 1 are guided. Since this implies that

$$\genfrac{}{}{0pt}{}{k_x}{k_y} \ll k_z\,, \tag{43}$$

the field components E_x in equation (38) can be neglected.

Now we match the remaining tangential components along the edges of region 1 and from equation (38) we obtain

$$\tan\left(k_y \frac{b}{2} \pm \beta\right) = i\,\frac{n_1^2}{n_{\genfrac{}{}{0pt}{}{2}{4}}^2}\,\frac{k_{y\genfrac{}{}{0pt}{}{2}{4}}}{k_y}. \tag{44}$$

$$\tan\left[k_x \begin{Bmatrix} \dfrac{c}{2} \\ a + \dfrac{c}{2} \end{Bmatrix} + \alpha\right] = i\,\frac{k_{x\genfrac{}{}{0pt}{}{5}{3}}}{k_x} \begin{bmatrix} ictn\left(k_{x5}\dfrac{c}{2} + \gamma\right) \\ 1 \end{bmatrix}. \tag{45}$$

Where there are two choices, the upper ones go together and the lower ones go together.

T. Li pointed out that each of these equations considered separately is the characteristic equation of a boundary value problem simpler than that of Fig. 14.[8, 9] Thus for a dielectric slab infinite in the x and z directions and refractive indexes as depicted in Fig. 15a, the characteristic equation for modes with no H_y component coincides with

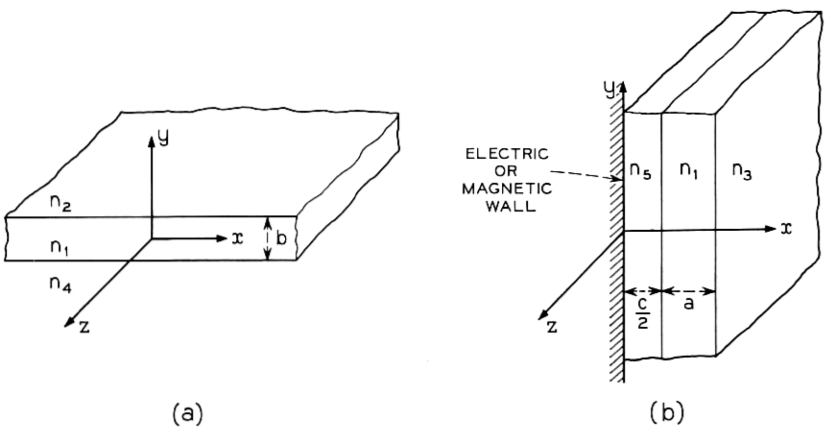


Fig. 15 — Dielectric slabs.

equation (44). Similarly, for two slabs infinite in the y and z directions and limited at $x = 0$ by an electric or magnetic short as in Fig. 15b, the characteristic equation for modes with $E_x = 0$ is equation (45).

A similar technique has been used by Schlosser and Unger to find the transmission properties of a rectangular dielectric guide immersed in another dielectric.[7] If the two guiding rods are so far apart that the coupling between them is a perturbation, then

$$| k_{x5} c | \gg 1 \tag{46}$$

and we can rewrite the characteristic equations (44) and (45) with the help of equations (39) and (46), making a and b explicit, as

$$k_y b = q\pi - \tan^{-1} \frac{n_2^2}{n_1^2} (k_y \eta_2) - \tan^{-1} \frac{n_4^2}{n_1^2} (k_y \eta_4) \tag{47}$$

$$k_x a = k_{x0} a \left[1 + \frac{2\xi_5}{a} \frac{\exp\left(-\frac{c}{\xi_5} - i2\gamma \right)}{1 + k_{x0}^2 \xi_5^2} \right] \tag{48}$$

where k_{x0} is the solution of

$$k_{x0} a = p\pi - \tan^{-1} k_{x0} \xi_3 - \tan^{-1} k_{x0} \xi_5 , \tag{49}$$

$$\eta_{\substack{2\\4}} = \frac{1}{| k_{y\substack{2\\4}} |} = \frac{1}{\left[\left(\frac{\pi}{A_{\substack{2\\4}}} \right)^2 - k_y^2 \right]^{\frac{1}{2}}} \tag{50}$$

$$\xi_{\substack{3\\5}} = \frac{1}{| k_{x\substack{3\\5}} |} = \frac{1}{\left[\left(\frac{\pi}{A_{\substack{3\\5}}} \right)^2 - k_{x0}^2 \right]^{\frac{1}{2}}} \tag{51}$$

and

$$A_{2,3,4,5} = \frac{\pi}{(k_1^2 - k_{2,3,4,5}^2)^{\frac{1}{2}}} = \frac{\lambda}{(n_1^2 - n_{2,3,4,5}^2)^{\frac{1}{2}}}. \tag{52}$$

In the transcendental equations (47) to (49), p and q are the arbitrary integers characterizing the order of the propagating mode, and the $\tan^{-1}$ functions are to be taken in the first quadrant. The angles $k_x a$ and $k_y b$ measure the phase shift of any field component across the guiding rod in the x and y directions respectively, or in other words, the electrical width and height of each guide of the coupler. On the other hand, $k_{x0} a$ is the electrical width of each guide assuming no interaction between the guides, that is assuming $c \to \infty$.

Let us find the physical significance of $\eta_{2,4}$ and $\xi_{3,5}$. The amplitude of each field component in medium 2 (Fig 14) decreases exponentially along y. It decays by $1/e$ in a distance η_2 given by equation (50). Similarly η_4, ξ_3, and ξ_5 measure the "penetration depth" of the field components in media 4, 3, and 5, respectively.

The propagation constant along z for each mode of the coupler is, according to equations (39), (40), and (41),

$$k_z = (k_1^2 - k_x^2 - k_y^2)^{\frac{1}{2}}. \tag{53}$$

With the help of equation (48), the slightly different propagation constants of the symmetric ($\gamma = 90°$) and antisymmetric modes ($\gamma = 0$) are

$$\left.\begin{matrix} k_{zs} \\ \\ k_{za} \end{matrix}\right\} = k_{z0}\left[1 \pm 2\frac{k_{x0}^2}{k_{z0}^2}\frac{\xi_5}{a}\frac{\exp(-c/\xi_5)}{1 + k_{x0}^2\xi_5^2}\right]. \tag{54}$$

In this expression

$$k_{z0} = (k_1^2 - k_{x0}^2 - k_y^2)^{\frac{1}{2}} \tag{55}$$

is the propagation constant of the E_{pq}^y mode of a single guide ($c \to \infty$).

The coupling coefficient K between the two guides and the length L necessary for complete transfer of power from one to the other are related to the propagation constants k_{zs} and k_{za} by[12]

$$\begin{aligned} -iK &= \frac{\pi}{2L} = \frac{k_{zs} - k_{za}}{2} = 2\frac{k_{x0}^2}{k_{z0}}\frac{\xi_5}{a}\frac{\exp(-c/\xi_5)}{1 + k_{x0}^2\xi_5^2} \\ &= \frac{2}{\pi}\frac{A_5 k_{x0}^2}{a k_{z0}}\left[1 - \left(\frac{k_{x0}A_5}{\pi}\right)^2\right]^{\frac{1}{2}} \exp\left\{-\frac{\pi c}{A_5}\left[1 - \left(\frac{k_{x0}A_5}{\pi}\right)^2\right]^{\frac{1}{2}}\right\}. \end{aligned} \tag{56}$$

As expected, the coupling increases exponentially both by decreasing c and by increasing the penetration depth ξ_5 in medium 5.

All these formulas contain either k_{x0} or k_y, which are solutions of the transcendental equations (47) and (49). For well-guided modes, most of the power travels within medium 1 and consequently

$$\left(\frac{k_{x0}A_{\substack{3\\5}}}{\pi}\right)^2 \ll 1 \tag{57}$$

and

$$\left(\frac{k_y A_{\substack{2\\4}}}{\pi}\right)^2 \ll 1. \tag{58}$$

It is possible then to solve those transcendental equations in a closed though approximate form by expanding the $\tan^{-1}$ functions in power of those small quantities and keeping the first two terms of the expansions. The explicit solutions of equations (47), (49), (50), (51), (55), and (56) are given in Section III.

A.2 E^x_{pq} *Modes: Polarization in the x Direction*

The field components and propagation constants can be derived from those in Section A.1 by changing E to H and μ to $-\epsilon$, and vice versa. Except for their polarizations, the E^x_{pq} and E^y_{pq} modes are very similar and have comparable propagation constants. Using boldface type to distinguish the symbols corresponding to E^x_{pq} modes, from equations (56), (55), (47), (49), (50), and (51), we obtain

$$-i\mathbf{K} = \frac{\pi}{2\mathbf{L}} = 2\frac{\mathbf{k}_{x0}^2}{\mathbf{k}_{z0}}\frac{\xi_5}{a}\frac{\exp(-c/\xi_5)}{1+(\mathbf{k}_{x0}\xi_5)^2} \tag{59}$$

where

$$\mathbf{k}_{z0} = (k_1^2 - \mathbf{k}_{x0}^2 - \mathbf{k}_y^2)^{\frac{1}{2}} \tag{60}$$

and $\mathbf{k}_{x0}$ and $\mathbf{k}_y$ are solutions of the transcendental equations

$$\mathbf{k}_y b = q\pi - \tan^{-1}\mathbf{k}_y\mathbf{n}_2 - \tan^{-1}\mathbf{k}_y\mathbf{n}_4 \tag{61}$$

and

$$\mathbf{k}_{x0}a = p\pi - \tan^{-1}\frac{n_3^2}{n_1^2}\mathbf{k}_{x0}\xi_3 - \tan^{-1}\frac{n_5^2}{n_1^2}\mathbf{k}_{x0}\xi_5 \tag{62}$$

in which

$$\mathbf{n}_{\substack{2\\4}} = \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{2\\4}}}\right)^2 - \mathbf{k}_y^2\right]^{\frac{1}{2}}} \tag{63}$$

and

$$\xi_{\substack{3\\5}} = \frac{1}{\left[\left(\dfrac{\pi}{A_{\substack{3\\5}}}\right)^2 - \mathbf{k}_{x0}^2\right]^{\frac{1}{2}}} \tag{64}$$

As in Section A.1, the transcendental equations (61) and (62) can be solved in closed, though approximate, form provided that

$$\left(\frac{\mathbf{k}_{x0}A_{\substack{3\\5}}}{\pi}\right)^2 \ll 1 \tag{65}$$

and

$$\left(\frac{\mathbf{k}_y A_{24}}{\pi}\right)^2 \ll 1. \qquad (66)$$

The explicit results are given in Section III.

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