WINDOWING AND FILTERING: FIR AND IIR

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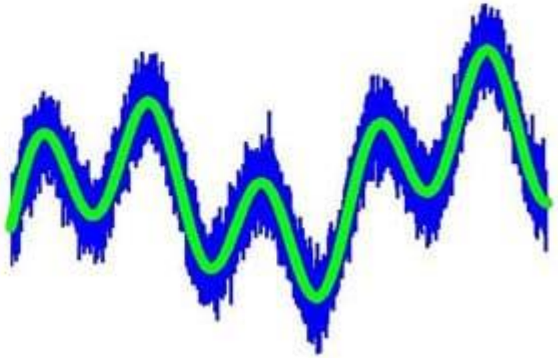
Reading: Chapters 9, 10 by Proakis and Monolakis

WHY TO FILTER? WINDOWS AND WINDOWING

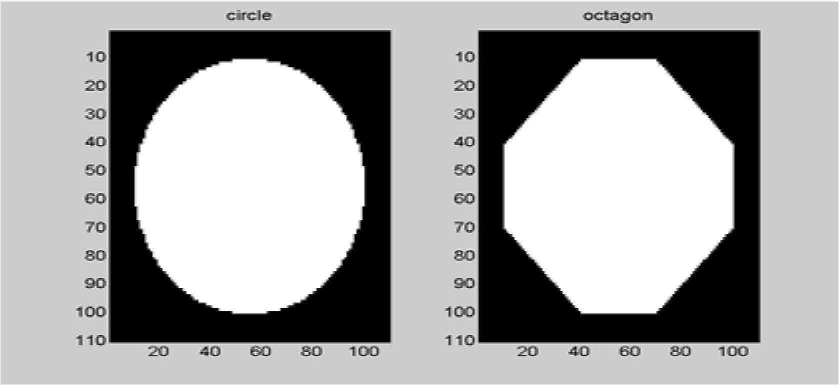
- In many DSP applications, very long signal samples must be processed.
- To practically manage the data, localized processing is applied to a subset of samples.
- This subset is often generated through the process of windowing.
- A straightforward or naive approach is to window by simply ignoring all points before a certain time instant and after a certain time instant.
- This amounts to what we call a rectangular window.

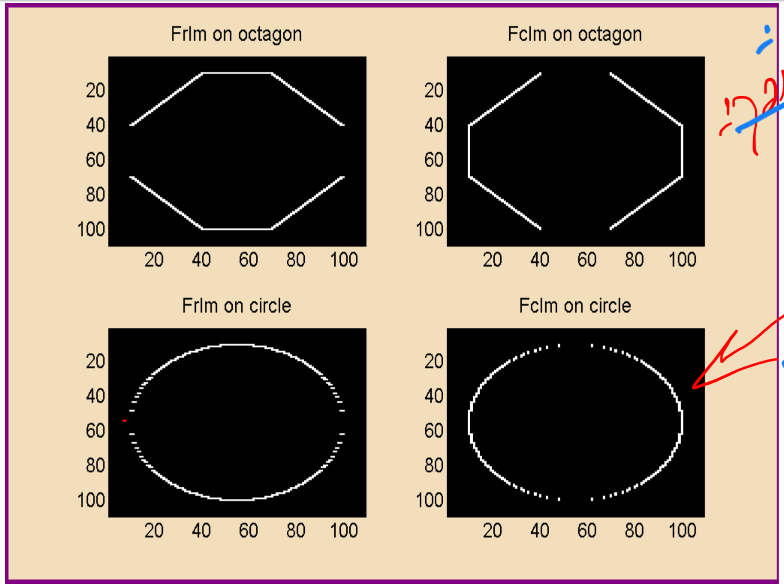
Filtering:

Extracting What We <u>Want</u> from What We <u>Have</u>

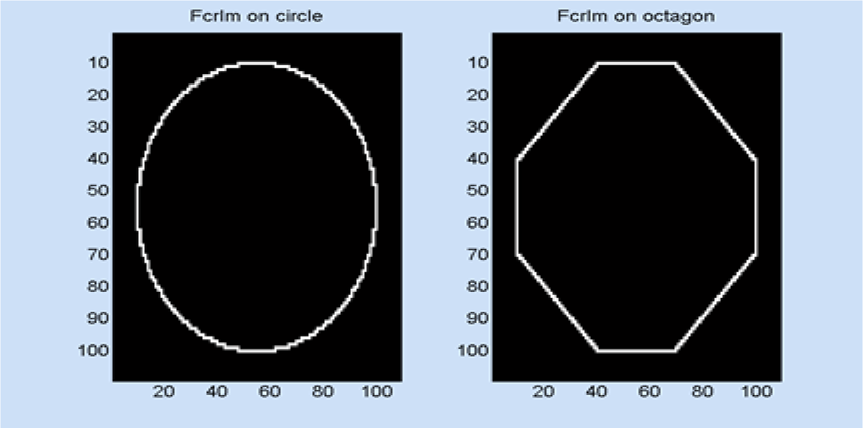


FILTERING EXAMPLE: IDENTIFICATION OF CONTOURS



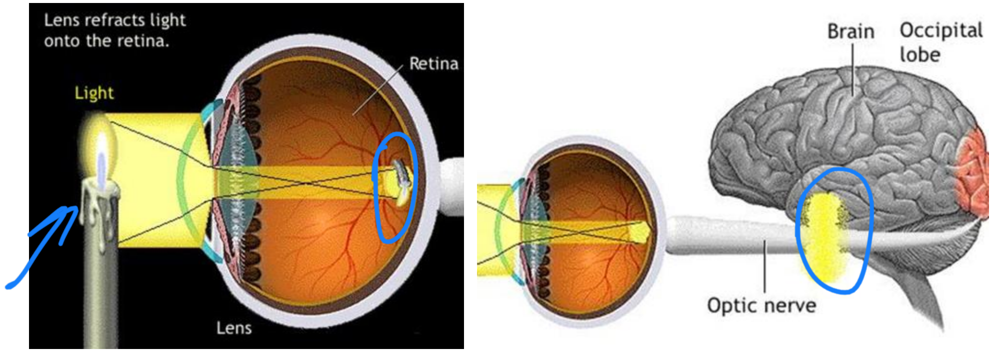


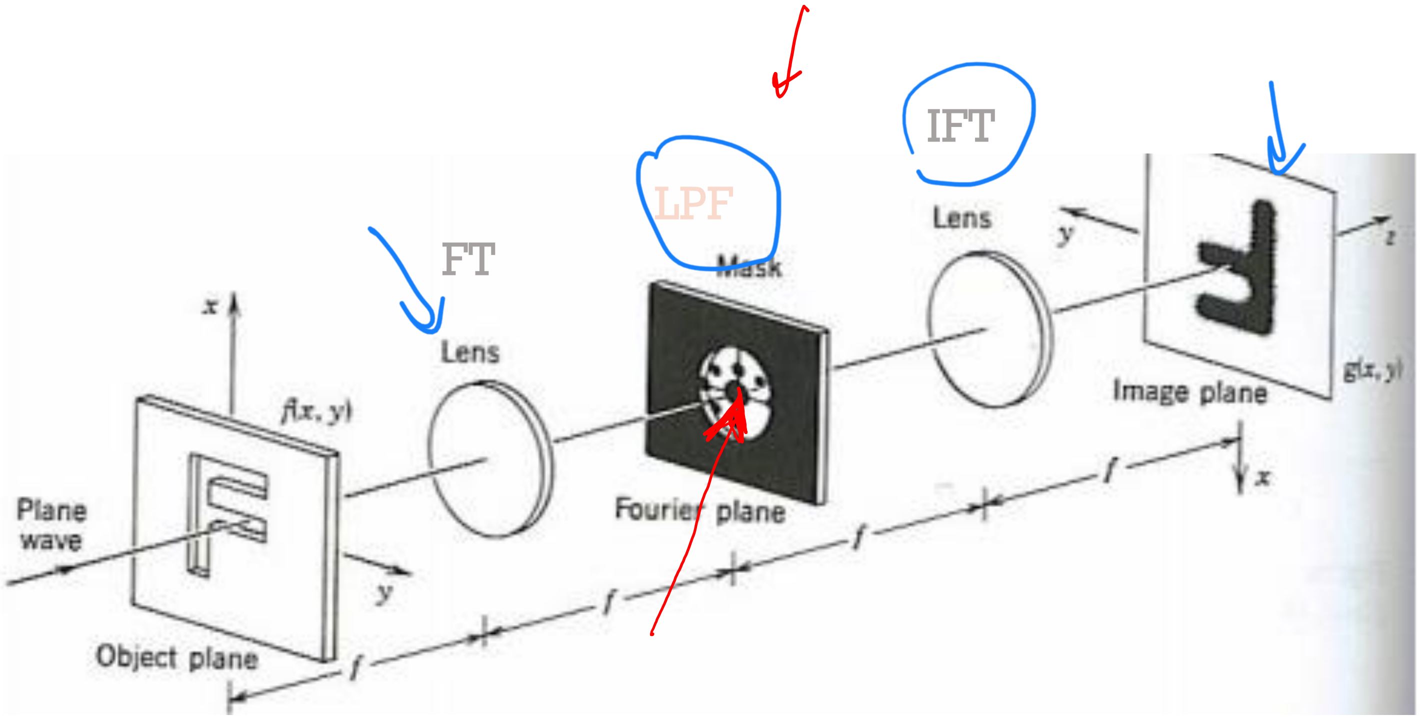
דוגמא: זיהוי קווי מתאר בעזרת פילטרים



שימוש במסננים $F_c = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $F_r = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ עבור כל אחת מהתמונות

ANALOG FILTERING: EXAMPLE

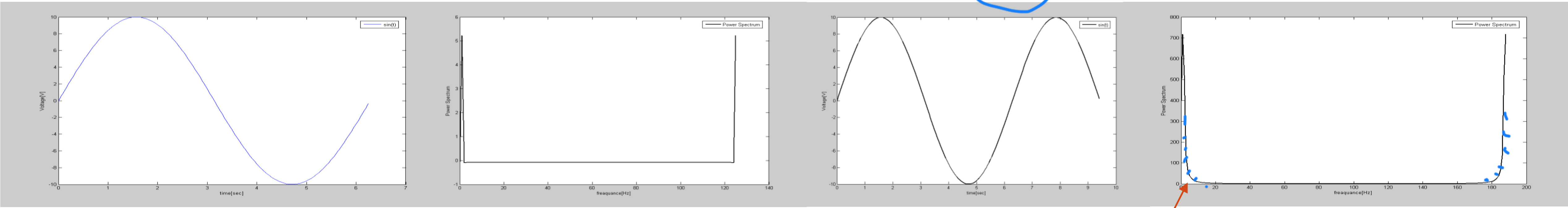




WHY TO FILTER? WINDOWS AND WINDOWING

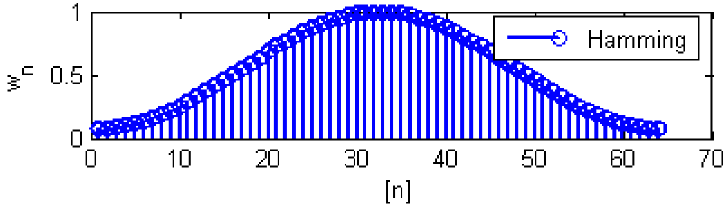
- For instance, sine signal in finite interval of 2π
- DFT of that appears as two delta functions
- However, if the sine signal has been truncated in interval $[0:2\pi]$, in the DFT of that signal appears 'spectral leakage' דליפה ספקטראלת that can be removed via windowing (filtering)

$x(t)$ $[0:2\pi]$ DFT $x(t)$ $[0:3\pi]$ DFT

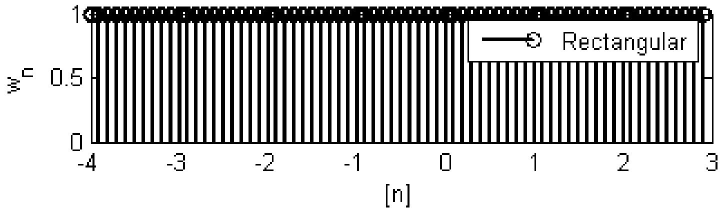


spectral leakage

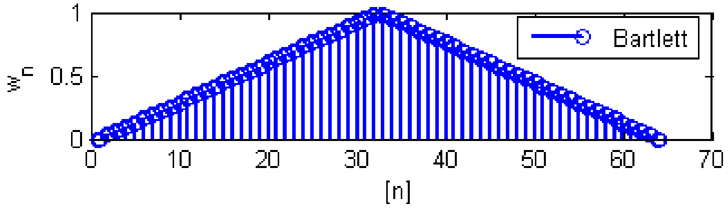
WINDOWS IN TIME: EXAMPLES



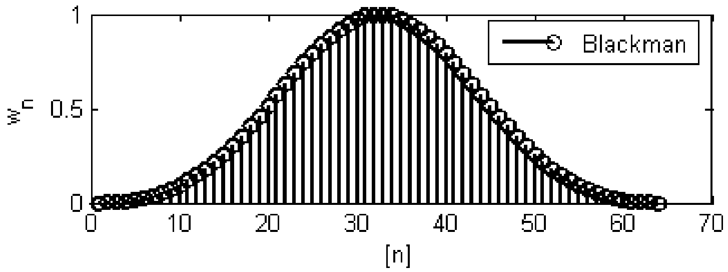
$$w_{\text{Hamming}}[n] = 0.54 - 0.46\cos\frac{2\pi n}{N} \qquad 0 \le n \le N$$



$$w_{\text{rec}}[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$$



$$w_{Bart}[n] = \begin{cases} \dfrac{n}{N/2} & 0 \le n \le \dfrac{N}{2} \\ W(N-1) & \dfrac{N}{2} \le n \le N-1 \end{cases}$$

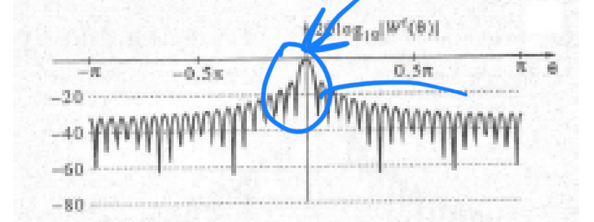
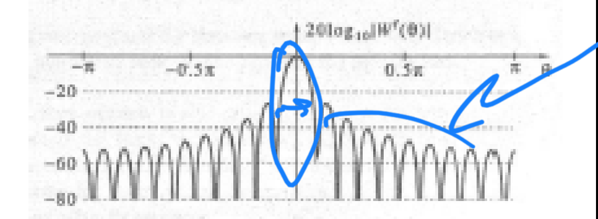
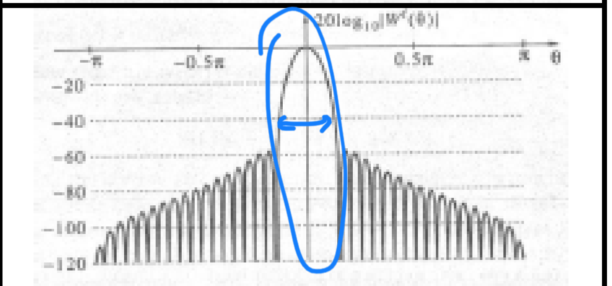


$$w_{\text{Blackman}}[n] = 0.42 - 0.5\cos\frac{2\pi n}{N} + 0.08\cos\frac{4\pi n}{N}$$

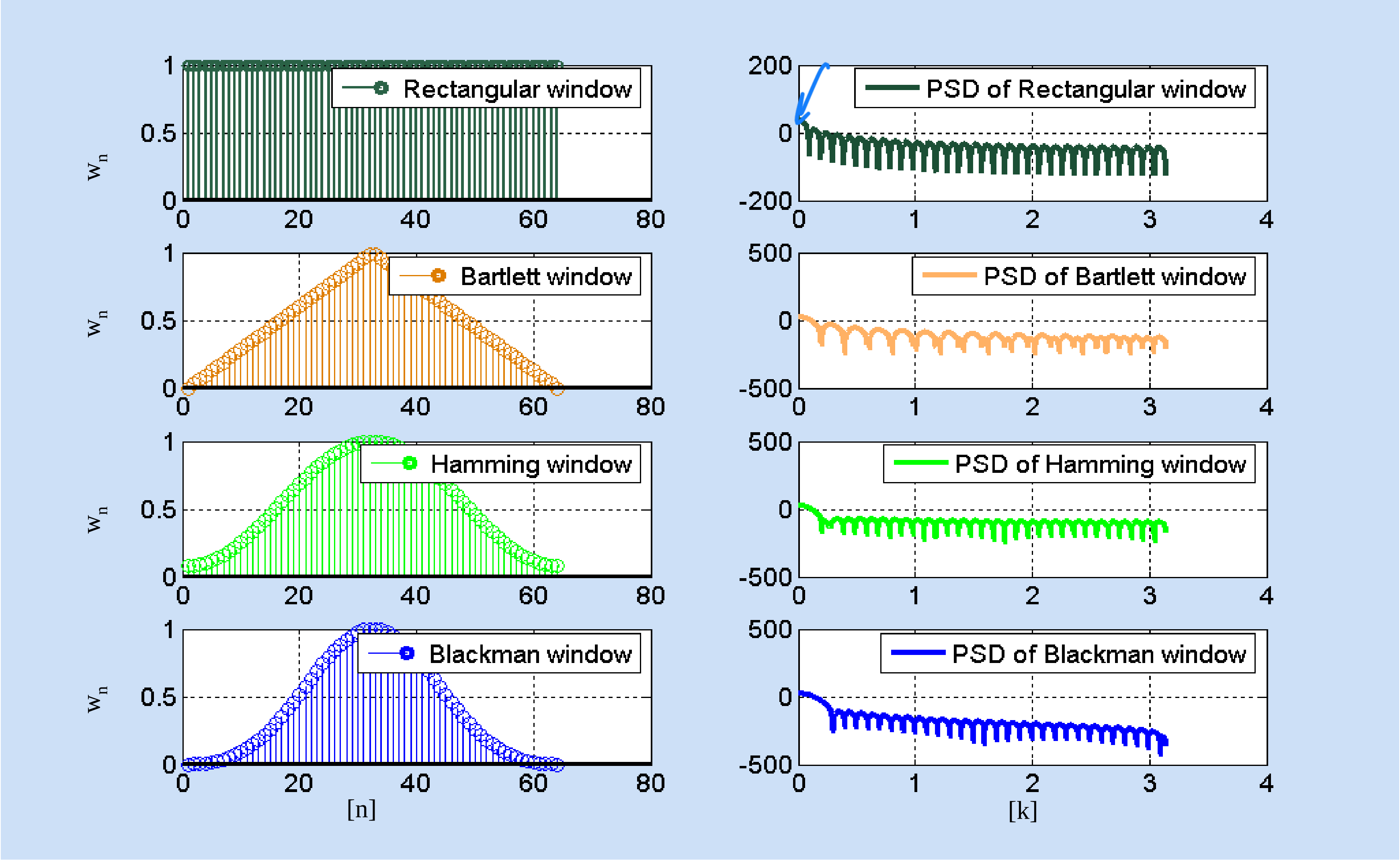
WINDOWS IN FREQUENCY: EXAMPLES

main lobe

side lobes

שם החלון	ספקטרום	המחשה
Rectangular	$W_{Rec}(\theta)=\dfrac{\sin\left(\frac{N}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}\cdot e^{-j\frac{N-1}{2}\theta}$	
Bartlett	$W_{Bart}(\theta)=\dfrac{2}{N}\left(\dfrac{\sin\frac{N}{4}\theta}{\sin\frac{1}{2}\theta}\right)^2\cdot e^{-j\left(\frac{N}{2}-1\right)\theta}$	
Hamming	$W_{Ham}(\theta)=0.54\cdot W_{Rec}(\theta)-$ $-0.23\cdot\left[W_{Rec}\left(\theta+\frac{2\pi}{N}\right)+W_{Rec}\left(\theta+\frac{2\pi}{N}\right)\right]$	
Blackman	$W_{Black}(\theta)=0.42\cdot W_{Rec}(\theta)+$ $+0.25\cdot\left[W_{Rec}\left(\theta+\frac{2\pi}{N}\right)+W_{Rec}\left(\theta-\frac{2\pi}{N}\right)\right]+$ $+0.04\cdot\left[W_{Rec}\left(\theta+\frac{4\pi}{N}\right)+W_{Rec}\left(\theta-\frac{4\pi}{N}\right)\right]$	

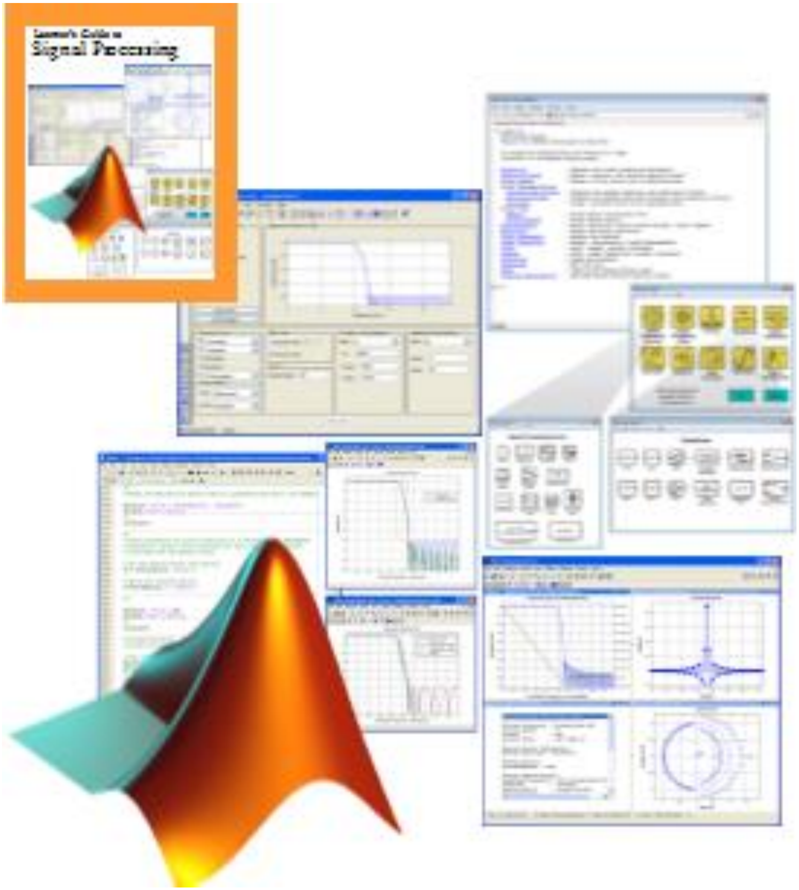
WINDOWS IN TIME VS FREQUENCY: EXAMPLES



שימוש בחלונות באורך של 64 דגימות תוך ריפוד ב עד לאורך של 256.

ENGINEERING CONNECTION

- Signal/Image processing: speech, image and video processing.
- Optoelectronics: filters are used to alter the colors of light by blocking certain ranges of wavelengths (see Lecture 1).
- Chemistry: membranes are used to block impurities while letting other materials pass through.



CRITERIA TO COMPARE BETWEEN WINDOWS 1: EQUIVALENT NOISE BANDWIDTH (ENBW)

מדד להערכת רוחב פס, קריטריון זה נותן פרמטר להערכת כמות האנרגיה שזולגת לנקודות תדר סמוכה בגלל תופעת ה-Spectral leakage (דליפה ספקטרלית) הנובעת מכך שאנו לוקחים קטע מוגבל בזמן ולא אינסופי. בשיטה זו מחשבים מלבן שהשטח שלו שווה בערכו לכמות האנרגיה של החלון. הקריטריון הינו רוחב המלבן, ולכן נותן הערכה לרוחב הסרט של ה-Spectral leakage. שטח המלבן הינו סה"כ האנרגיה של החלון ומחושב ע"י שיויון פרסיבל, הספק הרעש המצטבר של החלון מוגדר:

$$NoisePower = \frac{1}{N}\sum_{k=0}^{N-1}\left|W(k)\right|^2 = \sum_{k=0}^{N-1}\left|w(n)\right|^2$$

שטח המלבן נתון ע"י גובה^{x}רוחב, ולכן אם נחלק את שטח המלבן בגובה, נקבל את רוחב המלבן.
גובה המלבן נתון ע"י רכיב ה- DC של החלון:

$$W(0) = \sum_n w(n) \Rightarrow W^2(0) = \left[\sum_n w(n)\right]^2$$

ולכן רוחב המלבן נתון ע"י:

$$ENBW = \frac{\sum_n w^2(nT)}{[\sum_n w(nT)]^2}$$

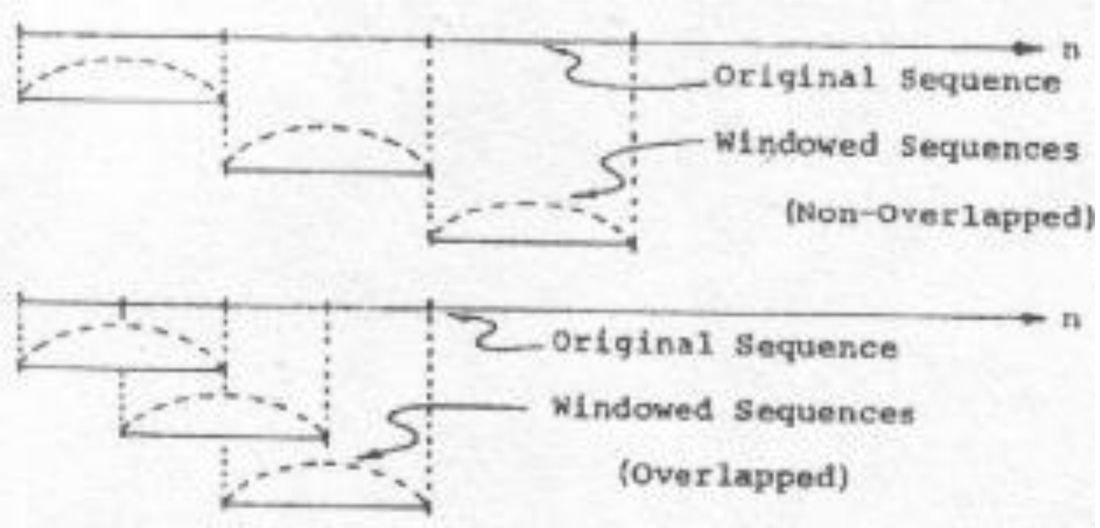
CRITERIA TO COMPARE BETWEEN WINDOWS 2: PROCESSING GAIN (PG)

- **processing Gain(PG)** קריטריון זה מגדיר את היחס בין האות לרעש במוצא s_o/N_o לבין היחס בין האות לרעש בכניסה s_{in}/N_{in}, והוא נתון ע"י הביטוי:

$$PG = \frac{s_o/N_o}{s_{in}/N_{in}} = \frac{\left[\sum_n w(nT)\right]^2}{\sum_n w^2(nT)} = \frac{1}{ENBW}$$

CRITERIA TO COMPARE BETWEEN WINDOWS 3: OVERLAP CORRELATION

<u>correlation</u>- מדובר על חפיפה בין קטעי הזמן לפי האיור הבא:



Harris: case of windows for harmonic analysis

כאשר מקדם הקורלציה נתון ע"י:

$$C(r)=\frac{\{\sum_{n=0}^{r\cdot N-1}(W(n)\cdot W(n+[1-r]N))\}}{\sum_{n=o}^{N-1}W^2(n)}$$

- זהו אחוז החפיפה בו תלויה הקורלציה.

החפיפה נדרשת משום שרוב החלונות נוטים לשאוף לערך אפס בקצוות, לכן יש צורך בחפיפה על מנת לקחת מחדש את המידע שהונחת בקצוות החלון.
המקרים המקובלים הם חפיפיה של כ- 50% ו- 75% מאורך החלון.

CRITERIA TO COMPARE BETWEEN WINDOWS 4: SCALLOPING LOSS

$$\text{Scalloping loss} = \frac{\left|\sum_n w(nT) \cdot exp\left(-i\frac{\pi}{N}n\right)\right|}{\sum_n w(nT)} = \frac{\left|w\left(\frac{1}{2} \cdot \frac{w_s}{N}\right)\right|}{w(0)}$$

הקריטריון מוגדר ע"י היחס בין ההגבר (ההכפלה בין האות לחלון) בנקודה אשר נמצאת בין שתי נקודות דגימה של ה-DFT [הביטוי שרשום במונה] לבין ההגבר בנקודת הדגימה עצמה [הביטוי שרשום במכנה].

קיימים 2 קריטריונים נוספים והם:
ה) הגודל של אונות הצד ביחס לאונה המרכזית.
ו) הערך האסימפטוטי שאליו מתכנסות האמפליטודות של אונות הצד.

CRITERIA TO COMPARE BETWEEN WINDOWS 5:

SIDE LOBES

Magnitude of the side lobes as compared to the main lobe

גודל של אונות צד ביחס לאונה המרכזית

אונה
מרכזית

אונות
צד

WINDOWS IN FREQUENCY: EXAMPLES

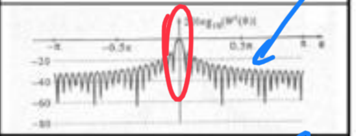
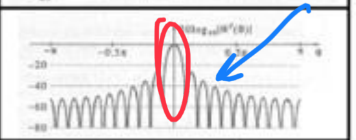
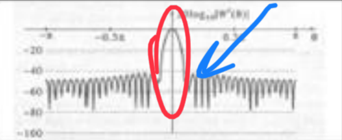
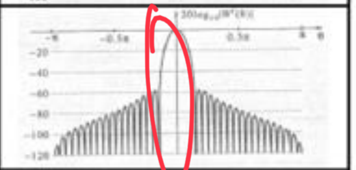
שם החלון	ספקטרום	המחשה
Rectangular	$W_{Rec}(\theta)=\frac{\sin\left(\frac{N}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}\cdot e^{-j\frac{N-1}{2}\theta}$	
Bartlett	$W_{Bart}(\theta)=\frac{2}{N}\left(\frac{\sin\frac{N}{4}\theta}{\sin\frac{1}{2}\theta}\right)^2\cdot e^{-j\left(\frac{N}{2}-1\right)\theta}$	
Hamming	$W_{Ham}(\theta)=0.54\cdot W_{Rec}(\theta)- -0.23\cdot\left[W_{Rec}\left(\theta+\frac{2\pi}{N}\right)+W_{Rec}\left(\theta+\frac{2\pi}{N}\right)\right]$	
Blackman	$W_{Black}(\theta)=0.42\cdot W_{Rec}(\theta)+ +0.25\cdot\left[W_{Rec}\left(\theta+\frac{2\pi}{N}\right)+W_{Rec}\left(\theta-\frac{2\pi}{N}\right)\right]+ +0.04\cdot\left[W_{Rec}\left(\theta+\frac{4\pi}{N}\right)+W_{Rec}\left(\theta-\frac{4\pi}{N}\right)\right]$	

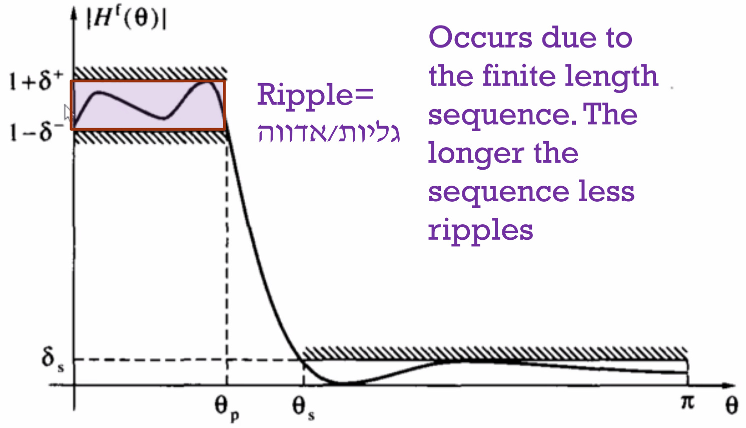
TABLE 10.2 Important Frequency-Domain Characteristics of Some Window Functions

Type of window	Approximate transition width of main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	−13
Bartlett	$8\pi/M$	−25
Hanning	$8\pi/M$	−31
Hamming	$8\pi/M$	−41
Blackman	$12\pi/M$	−57

CRITERIA TO COMPARE BETWEEN WINDOWS 6: ASYMPTOTIC VALUE OF MAIN LOBE AMPLITUDE

הערך האסימפטוטי שאליו מתכנסות האמפליטודות של אונות הצד.

FILTERS: CLASSIFICATION



Ripple= גליות/אדווה

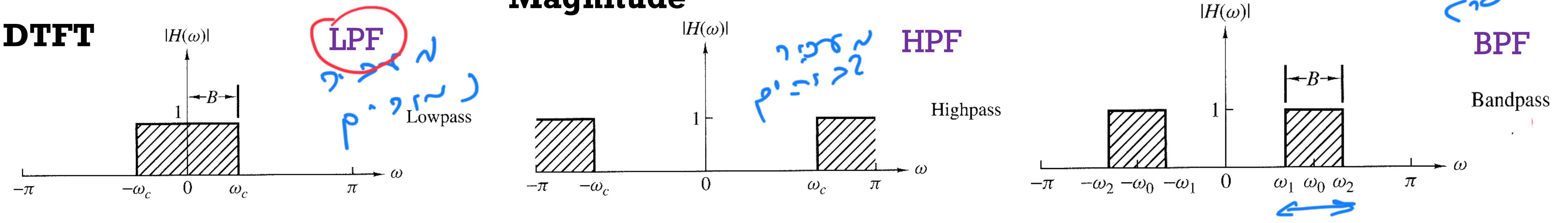
Occurs due to the finite length sequence. The longer the sequence less ripples

Figure 8.2 Specification of a low-pass filter.

- There are two kinds of filters:
- I) **Frequency selective** (see Proakis and Monolakis Ch. 10)– such as High Pass Filter (HPF), Low Pass Filter (LPF), Notch filters, Comb filters. – will be studied in DSP course. II) **Specific filters** – for specific implementation

Magnitude

DTFT



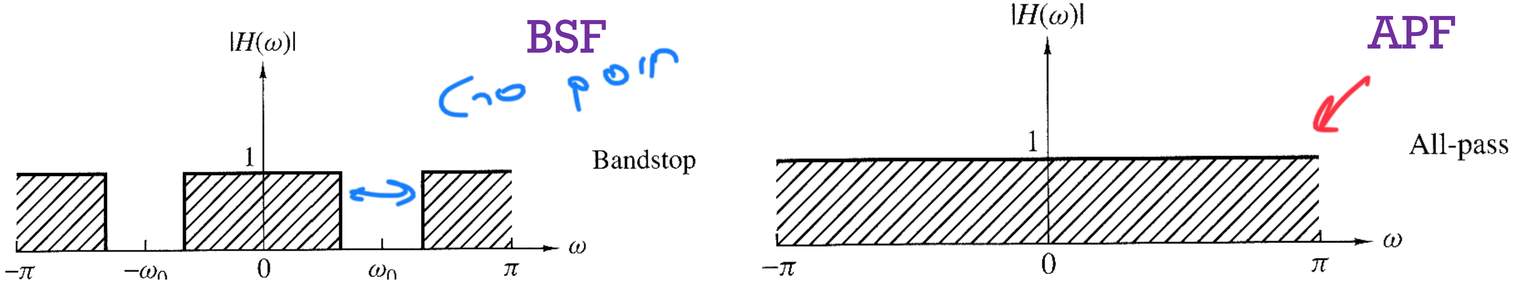
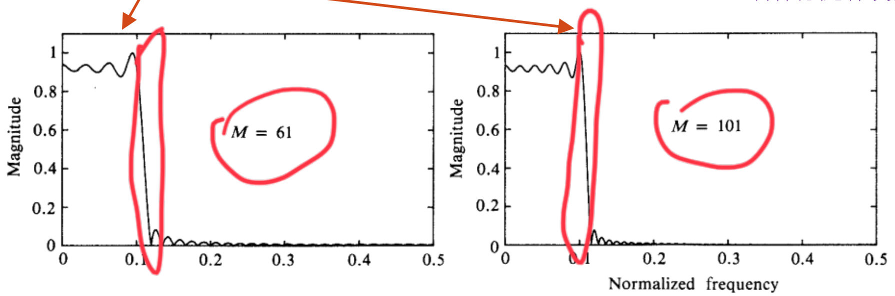


Figure 5.4.1
Magnitude responses for some ideal frequency-selective discrete-time filters.

Phase?

FILTERS: DEFINITION

Gibbs phenomenon



Ripple= גליות/אדווה

Occurs due to the finite length sequence. The longer the sequence less ripples

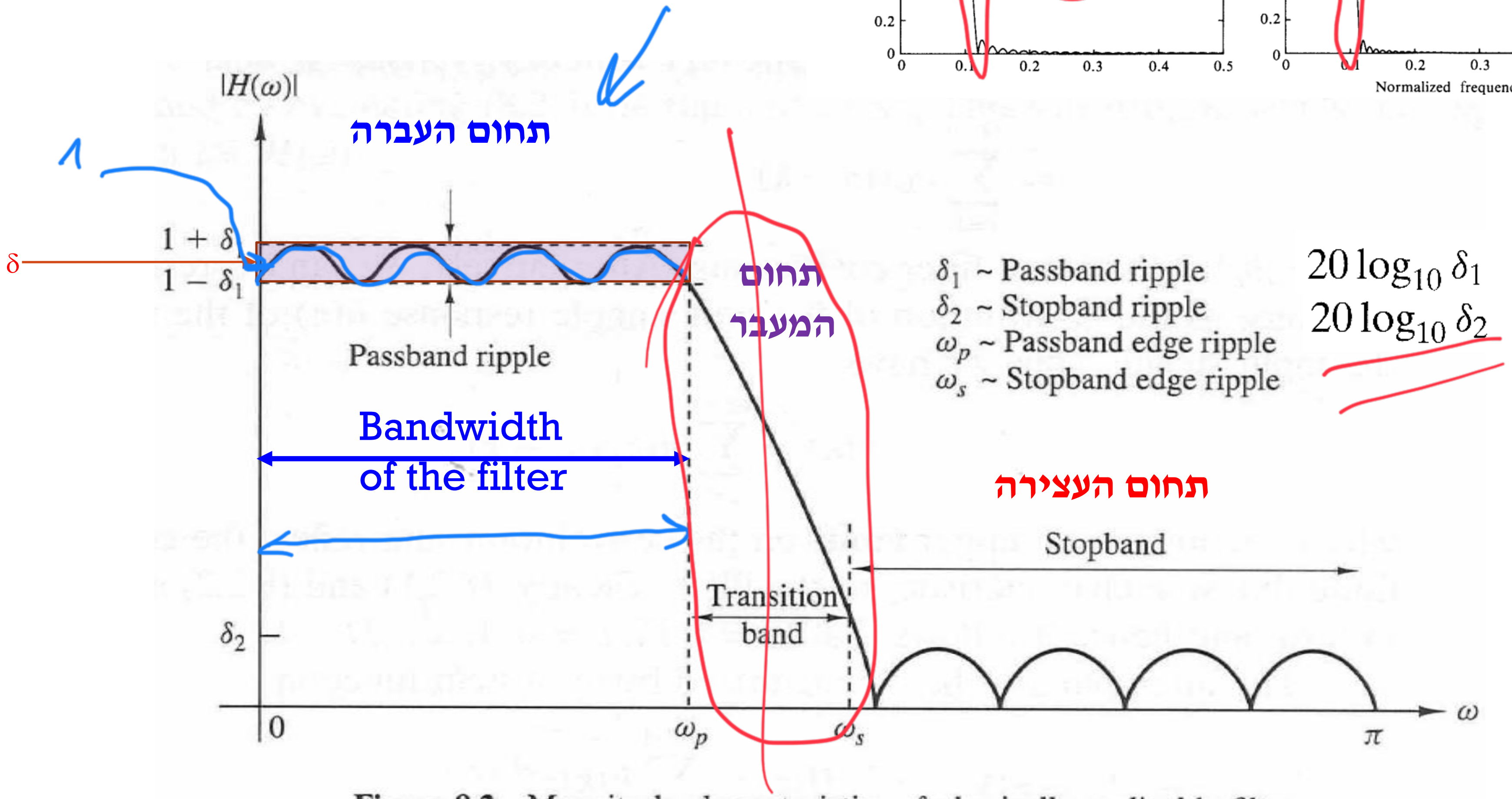
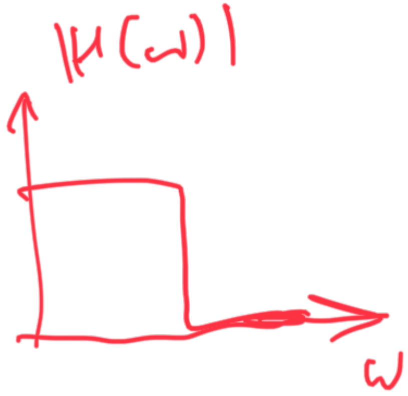


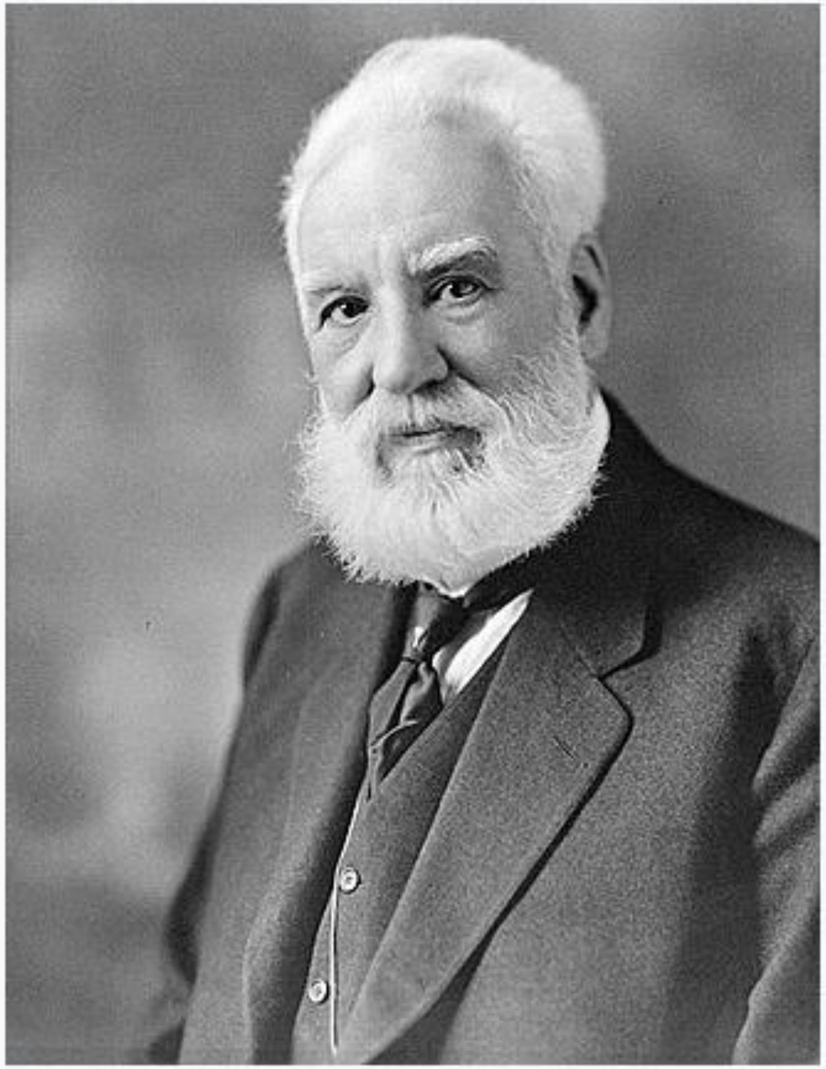
Figure 8.2 Magnitude characteristics of physically realizable filters.



RELATION BETWEEN BEL AND DECIBEL

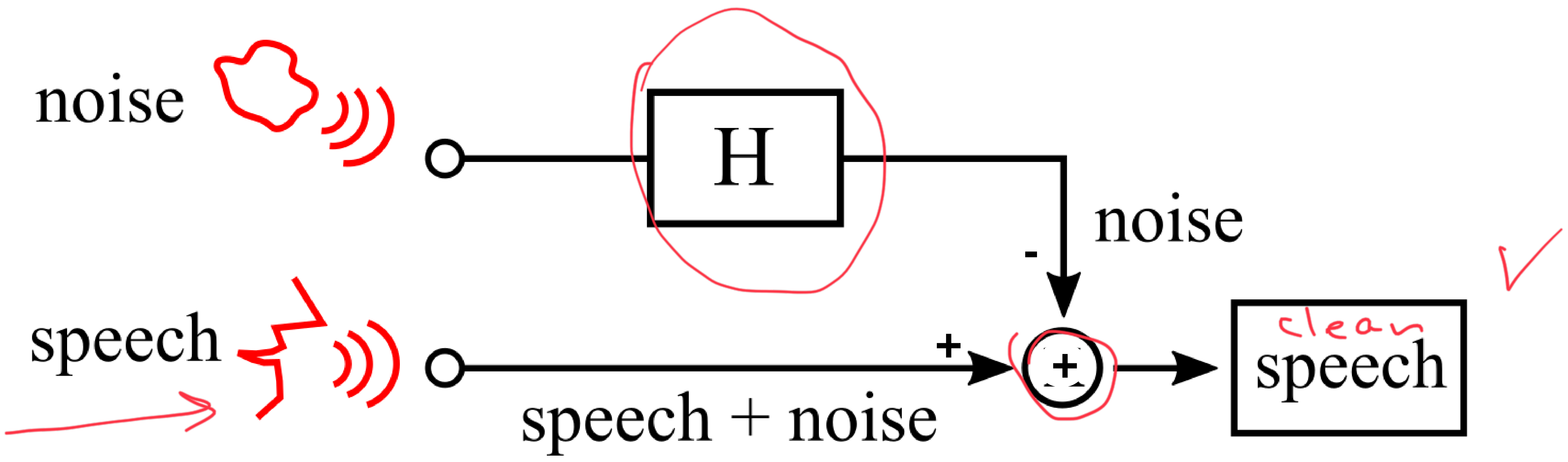
- The **decibel** (symbol: dB) is a relative unit of measurement equal to one tenth of a bel (B). The **bel** was named in honor of Alexander Graham **Bell**.
- Alexander Graham Bell was a Scottish-born inventor, scientist and engineer who is credited with patenting the first practical telephone, groundbreaking work in optical telecommunications.
- Bell Laboratories (9 Nobel prizes) was, and is, regarded by many as the premier research facility of its type, developing a wide range of revolutionary technologies, including radio astronomy, the transistor, the laser, information theory, the operating system Unix, the programming languages C and C++, solar cells, the charge-coupled device (CCD), and many other optical, wireless, and wired communications technologies and systems

Alexander Graham Bell



Bell c. 1917

Born Alexander Bell
March 3, 1847
Edinburgh, Scotland



FILTERS: DESIGN

- Design filters are studied in:
- I) **Frequency selective** filters: DSP course
- II) **Specific filters** – for specific implementation. For instance: filtering the noise from a noisy speech – studied in statistical signal processing of adaptive filter סינון אדפטיבי, סטטיסטי ועיבוד אותות

אילוצי מימוש

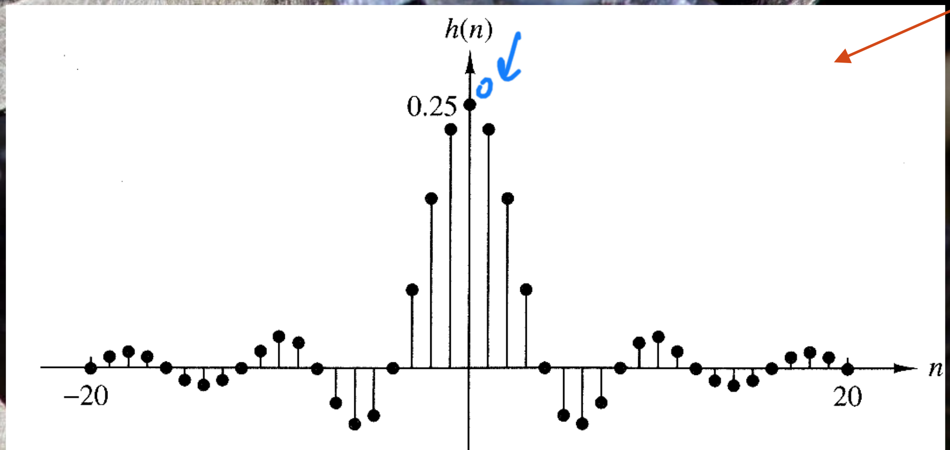
IMPLEMENTATION CONSIDERATIONS

- Let investigate the ideal filter of Low Pass Filter (LPF) kind

$$H(e^{j\theta}) = \begin{cases} 1 & |\theta| \le \theta_c \\ 0 & \theta_c \le |\theta| \le \pi \end{cases}$$ from this:

Impulse response

$$\rightarrow h[n] = \frac{\theta_c}{\pi}\frac{\sin(\theta_c n)}{\theta_c n}$$



Can we implement such a filter? No, since it is not causal. לא סיבתי. לא ניתן למימוש

סיבתיות

Causality – is the first condition to implement a filter.

In case of LPF, one cannot implement it since the response is not causal.

CAUSALITY

$$H^f(\theta) = H_R^f(\theta) + jH_I^f(\theta)$$
$$= |H^f(\theta)|e^{j\psi(\theta)}$$
$$A = \sqrt{\left(H_R^f(\theta)\right)^2 + \left(H_I^f(\theta)\right)^2}$$

אנרגיה סופית:

- 1) For h[n] of finite energy and casual the following theorem must be valid:

Paley-Wiener Theorem (1934) $\int_{-\pi}^{\pi} |\ln|H(e^{j\theta})|| \, d\theta < \infty$ $\quad |H| \neq \infty$

This condition is not valid if ∞, $|H(e^{j\theta})| = 0$ but $|H| \neq \infty$

because finite energy is given

Analysis of LPF is valid for ideal filters such as BPF, HPF. Since those filters also have rectangular window function in frequency

- 2) **Causality and the relation between H to H_I, H_R**

Imag Real

גודל פאזה -

Filter construction without conditions on H can allow to decide upon |H|, <H as we wish and so to define H_I, H_R as we wish

$$\frac{H_R^f(\theta)}{|H^f(\theta)|} = \sin(\psi)$$
$$\frac{H_I^f(\theta)}{|H^f(\theta)|} = \cos(\psi)$$
$$\frac{H_R^f(\theta)}{|H_I^f(\theta)|} = \tan(\psi)$$

For $h[n] \leftrightarrow H(e^{j\theta})$ we can show the **even** part of the **time response**:

$$h[n] = h_e[n] + h_o[n]; \quad h_e[n] = \frac{1}{2}[h[n] + h[-n]]; \; h_o[n] = \frac{1}{2}[h[n] - h[-n]]$$

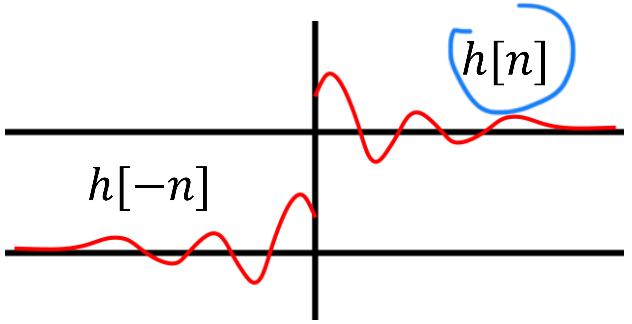
CAUSALITY AND THE RELATION BETWEEN H AND H_R, H_I

For $h[n] \leftrightarrow H(e^{j\theta})$ we can show the **even** part of **time response**:

$$h_e(n) = \frac{1}{2}[h[n] + h[-n]]$$

In general, one cannot reconstruct h[n] from $h_e[n]$. Can we reconstruct if $h[n]$ is casual? Yes, because $h[n]$ and $h[-n]$ overlap only in $n = 0$.

- $h[n] = 2h_e[n]u[n] - h_e[0]\delta[n]$ $n \geq 0$ ->
- לכן קיים קשר חד-חד ערכי בין
- $h_e[n] \leftrightarrow h[n]$ $h[n]$ is completely determined from $h_e[n]$



- ולאחר התמרה DTFT נקבל קשר חד-חד ערכי בין

$h_e(n) \overset{F}{\longleftrightarrow} H_R(\omega)$

$H(\omega) = H_R(\omega) + jH_I(\omega)$ →

$h_o(n) \overset{F}{\longleftrightarrow} H_I(\omega)$

$H_R(e^{j\theta}) \leftrightarrow H(e^{j\theta})$

real

Therefore, $H(e^{j\theta})$ is completely determined if we know $H_R(e^{j\theta})$. Equivalently, the magnitude and phase responses of a casual filter are interdependent and hence cannot be specified independently.

FILTERS IMPLEMENTATION: FIR AND IIR

FIR

FIR Definition:

תגובה להלם סופית

Finite Impulse Response (**FIR**) filter- is a filter with finite impulse response which fulfill the following equation:

כניסה

$$\underset{\text{מוצא}}{y[n]} = \sum_{l=0}^{L-1} \underset{\text{תגובה להלם}}{h[l]}\, x[n-l]$$

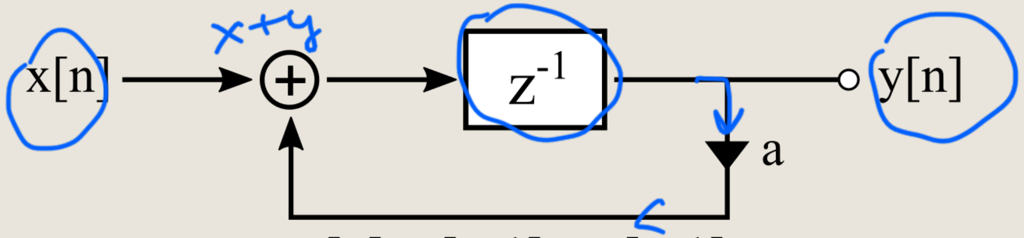
$$= \sum_{l=0}^{L-1} b_l\, x[n-l]$$

b_l is vector of filter's coefficients also in impulse response:

Note: **Finite Impulse Response** (FIR) is also named **Moving Average** (MA): average is due to the weighting of x. Moving because of the convolution window.

FILTERS IMPLEMENTATION: FIR AND IIR

IIR



y[n]=x[n-1]+ay[n-1]

<u>IIR</u> <u>Definition</u>:

תגובה להלם אינסופית

Infinite Impulse Response (IIR) filter- is a filter in which the output is dependent also on the output in past and fulfill the following:

כניסה

מוצא

$$y[\text{n}] = \underbrace{\sum_{l=0}^{L-1} b_l x[n-l]}_{\text{FIR}} - \sum_{m=1}^{M-1} a_m y[n-m]$$

משוב feed back

a_m, b_l filter's coefficients

One can present also the infinite impulse response due to the feedback of y.

Infinite Impulse Response (IIR) is also named Auto-Regressive Moving Average (ARMA) – self-feedback moving averaging

FREQUENCY RESPONSE OF FILTERS – VIA DTFT

תגובת התדר של מסננים באמצעות DTFT

DTFT transform – multiplication in $e^{-j\theta n}$ and summation on n

We will obtain:

$$Y(e^{j\theta}) = \sum_{l=0}^{L-1} b_l X(e^{j\theta}) e^{-j\theta l} - \sum_{m=1}^{M-1} a_m Y(e^{j\theta}) e^{-j\theta m}$$

$$Y(e^{j\theta}) \left[1 + \sum_{m=1}^{M-1} a_m e^{-j\theta m} \right] = X(e^{j\theta}) \sum_{l=0}^{L-1} b_l e^{-j\theta l}$$

And the frequency response is:

$$\boldsymbol{H(e^{j\theta}) = \frac{Y(e^{j\theta})}{X(e^{j\theta})} = \frac{\sum_{l=0}^{L-1} b_l e^{-j\theta l}}{\sum_{m=0}^{M-1} a_m e^{-j\theta m}}}$$

while $a_0 = 1$

SURVEY: A FILTER PROPERTIES

- EasyPolls:

Which one of the following properties is fulfilled by each filter defined by y[n]=sum m=(-1000) to 1000. b[m]x[n-m], b is finite

איזו מהתכונות הבאות

מתקיים ע"י כל מסנן המוגדר ע"י

$$y(n) = \sum_{m=-1000}^{1000} b(m)\,x(n-m)$$

כאשר b סופי.

- ✓ O BIBO stability
- ✗ O causality
- ✓ O time-invariant — קבועים בזמן
- לא הפיכים O inversible — הפיכים
- ✓ O more than one property is correct

results vote

EXAMPLE: FIR

Given: $b_l = \{1, 3, 2\}$, $l = 0,1,2$

So impulse response would be: $h[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2]$

Calculate z transform: $H(z) = 1 + 3z^{-1} + 2z^{-2}$

Delay in 1

present as positive power of z

$$= \frac{z^2 + 3z + 2}{z^2}$$

$$= \frac{(z+1)(z+2)}{z^2}$$

$z+1=0$

$\boxed{z=-1}$

$z+2=0$

$\boxed{z=-2}$

- Is it stable? Yes, since $h[n]$ is right and the poles are inside the unite circle.

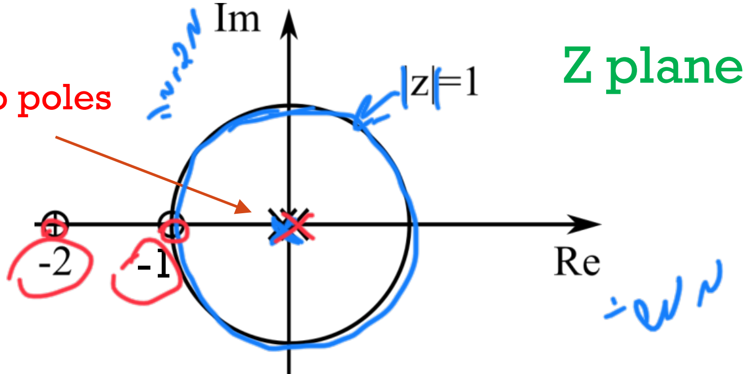
Is it always stable? Yes. The poles are always at $z = 0$.

Advantage: FIR is always stable: poles inside the unit circle

If the system is righteous מערכת ימנית

Disadvantage: poles are always at $z = 0$

this limits the design of the filter



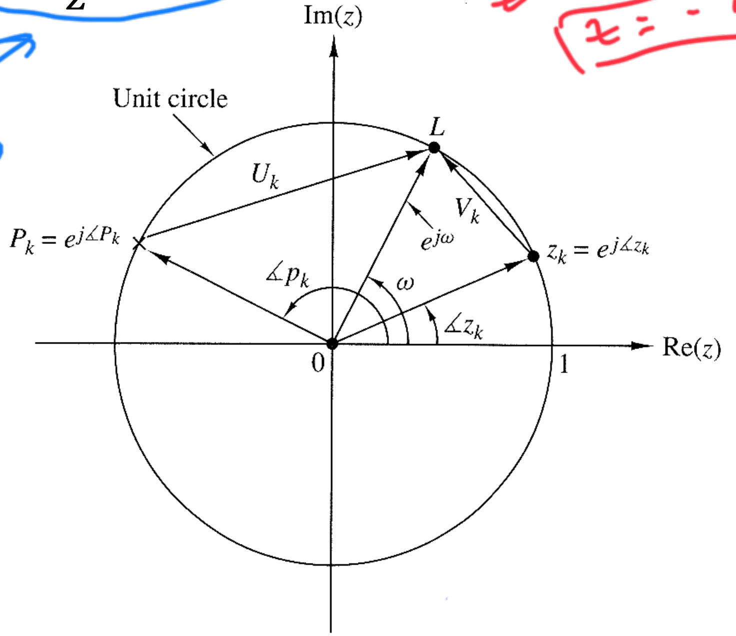


Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

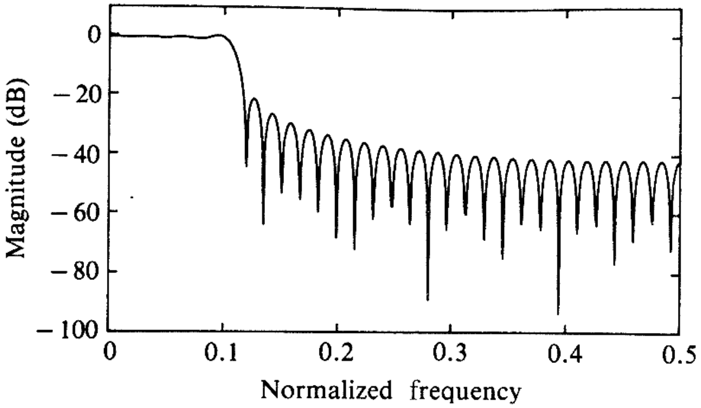


Figure 10.2.8
Lowpass FIR filter designed with rectangular window ($M = 61$).

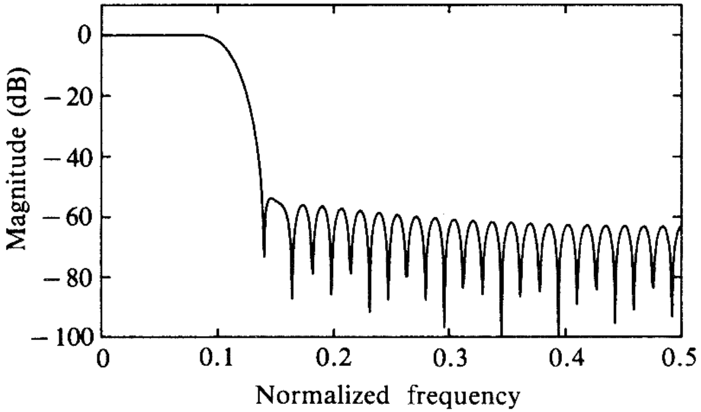


Figure 10.2.9
Lowpass FIR filter designed with Hamming window ($M = 61$).

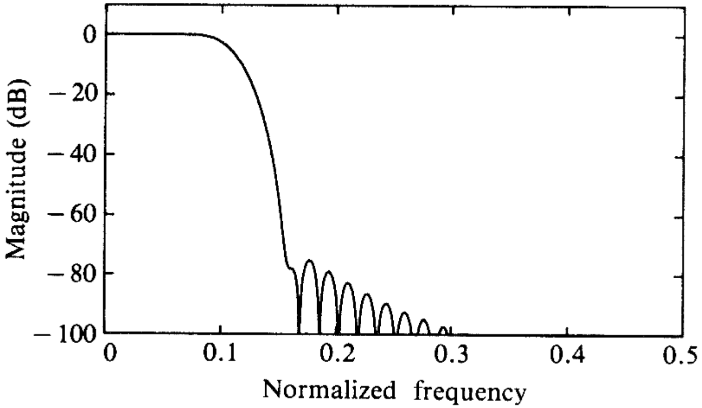


Figure 10.2.10
Lowpass FIR filter designed with Blackman window ($M = 61$).

LOWPASS FILTER DESIGN WITH A WINDOW

To alleviate the presence of large oscillations in both the passband and the stopband, we should use a window function that contains a taper and decays toward zero gradually, instead of abruptly, as it occurs in a rectangular window. Figures 10.2.8 through 10.2.11 illustrate the frequency response of the resulting filter when some

EXAMPLE: FIR SUMMARY

- Is it stable? Yes, since $h[n]$ is right and the poles are inside the unite circle.
  Is it always stable? Yes. The poles are always at $z = 0$.

Advantage: FIR is always stable

Disadvantage: poles are always at $z = 0$ this limits the design of the filter

מוצא

כניסה

$$\boxed{y[\mathrm{n}] = \sum_{l=0}^{L-1} b_l x[n-l]}$$

מקדמי
המסנן

EXAMPLE: IIR

Z plane

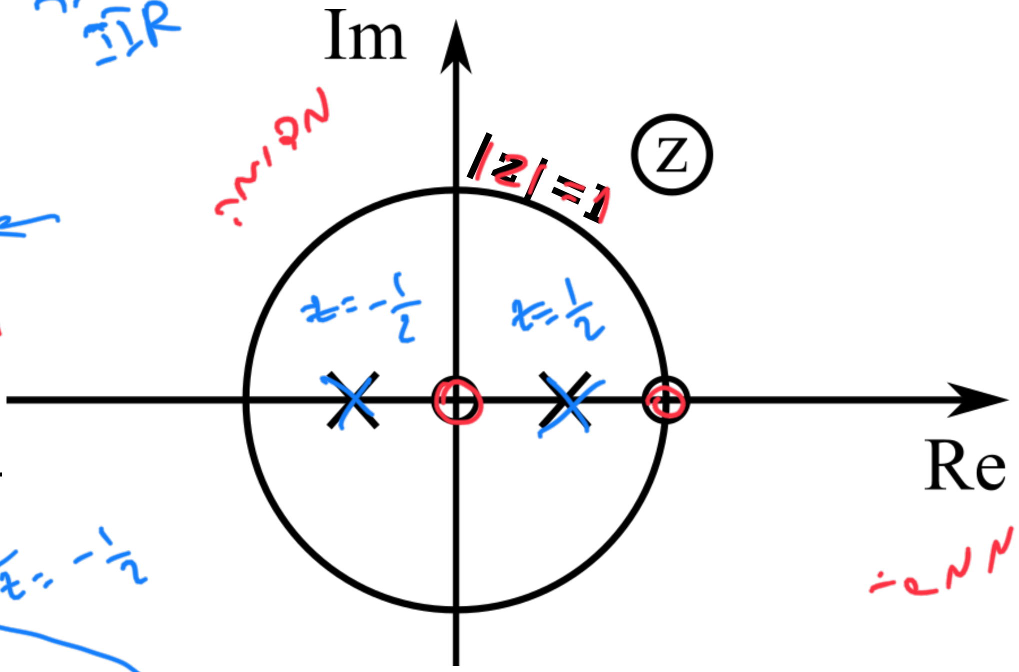
- $y[n] = x[n] - x[n-1] + \frac{1}{4}y[n-2]$

Z transform
- $Y(z) = X(z) - z^{-1}X(z) + \frac{1}{4}z^{-2}Y(z)$
- $Y(z)[1 - \frac{1}{4}z^{-2}] = X(z)[1 - z^{-1}]$
- $H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{1}{4}z^{-2}} = \frac{z^2 - z}{z^2 - \frac{1}{4}} = \frac{z(z-1)}{(z-\frac{1}{2})(z+\frac{1}{2})}$

$z=0$ $z=1$ $z=\frac{1}{2}$ $z=-\frac{1}{2}$

Is it stable? Yes

Zeros and Poles can be in each place on z



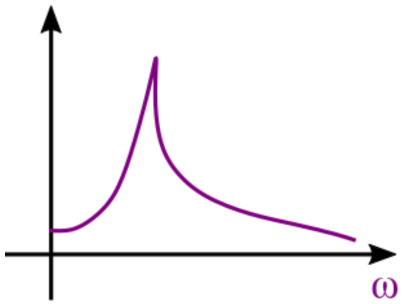
EXAMPLE: IIR SUMMARY

Advantage: flexibility in choosing poles

Disadvantage: can be unstable

$$\boxed{y(n) = \underbrace{\sum_{l=0}^{L-1} b_l x[n-l]}_{\text{FIR}} - \sum_{m=1}^{M-1} a_m \underbrace{y[n-m]}}$$

COMPARISON: FIR VS. IIR



Property	FIR	IIR
I Computational complexity יעילות חישובית	X	V
II BIBO stability	Always stable, V	לא תמיד יציב Not always stable, ?
III Linear Phase, delay = no distortion	V V	Not linear, can be approximated
IV Error accumulation לא הצטברות השגיאות	V	Recursive calculation, X
V Integration in optimization problems (noise filtering, needed response) שילוב בבעיות אופטימיזציה סינון רעש, תגובה נדרשת	Cost function convex, V	Cost function non-convex, ~~V~~ X