# DISCRETE FOURIER TRANSFORM (DFT): LONG SEQUENCES, MATRIX NOTATION

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Reading: Chapter 4 by B. Porat

# DFT PROPERTY: SYMMETRY

- Let assume signal $x[n]$ as real

Proof that $X[k] = X^*[N-k]$

Proof:

$$X^*[N-k] = \sum_{n=0}^{N-1} \left[x[n]e^{-j(N-k)\frac{2\pi}{N}n}\right]^*$$

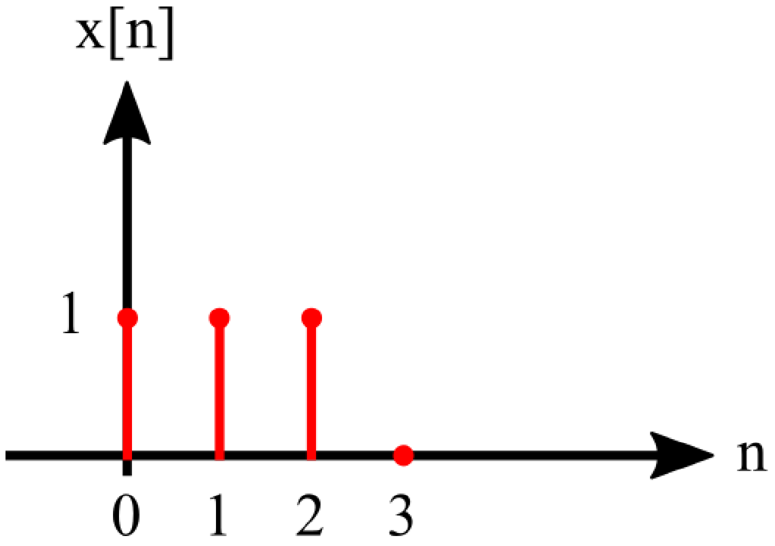
$$= \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}e^{jN\frac{2\pi}{N}n} = 1$$

$$= X[k]$$

Meaning:

1) One can calculate only samples $k = 0 \dots N/2$ ($N$ is even) or $k = 0 \dots (N-1)/2$ ($N$ is odd)
2) If $N$ is even then $X[N/2]$ is real.

# EXAMPLE



- Let $x[n]$ a signal composed on ones of Length $N = 4$
- This is discrete square wave

1) Calculate the DFT $X[k]$. Note $X[k]$ is complex
2) In Personal Computer (PC) there will be two vectors $X_R$ and $jX_I$ as two separated vectors/arrays. What is the inverse transform (IDFT) of each?

# EXAMPLE: SOLUTION – DFT CALCULATION

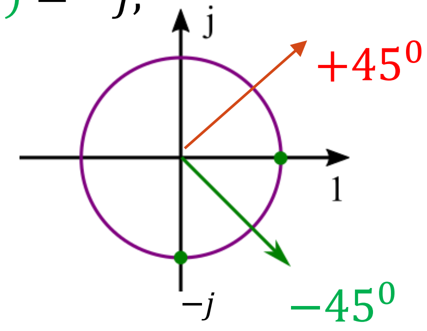
1) From the definition of DFT:

$$X[k] = \sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}nk} = \sum_{n=0}^{2} e^{-j\frac{2\pi}{4}nk} = \frac{1-e^{-j\frac{2\pi}{4}k3}}{1-e^{-j\frac{2\pi}{4}k}} =$$

$$= \frac{1-e^{-jk\frac{3\pi}{2}}}{1-e^{-j\frac{\pi}{2}k}}$$

$$X[0] = 3,\ sum\ of\ x[n] \longrightarrow X[k=0] = \sum_{n=0}^{N-1}\left(W_N^{kn}\right) = N \cdot \delta[k \bmod N]$$

$$X[1] = \frac{1-\overset{-\frac{3\pi}{2}}{j}}{\underset{-\frac{\pi}{2}}{1-(-j)}} \overset{graphically}{\rightarrow} \frac{-45^0}{+45^0} \overset{graphically}{\rightarrow} -45^0-(+45^0) = -j,$$



$$X[2] = \frac{1-\overset{-3\pi}{(-1)}}{1-\underset{-\pi}{(-1)}} = 1,$$

$$X[3] = X^*[\mathrm{N}-3] = \underset{X^*[1]}{X^*[4-3]} = -j^* = j$$

To save in Personal Computer (PC) there will be two vectors $X_R$ and $jX_I$ as two separate vectors/arrays:

$X_R[k] = \{3,0,1,0\}$ $jX_I[k] = j\{0,-1,0,1\}$

# EXAMPLE: SOLUTION – IDFT CALCULATION

2) Now we will apply the inverse transform on $X_R$ and $jX_I$

$$X[k] = X_R[k] + jX_I[k] \;\rightarrow\; X_R[k] = \{3,0,1,0\} \qquad jX_I[k] = \{0,-j,0,j\} = j\underbrace{\{0,-1,0,1\}}_{\text{Saved in PC this way}}$$

We now calculate inverse transforms for $X_R[k]$ and for $X_I[k]$:

$$0 \le n \le 3 \qquad x_1[n] = \frac{1}{4}\sum_{k=0}^{3} X_R[k]e^{j\frac{2\pi}{4}nk} =$$

$$= \frac{1}{4}\left[\underset{k=0}{3} + \underset{k=2}{e^{j\frac{2\pi}{4}n\cdot 2}}\right] = \frac{1}{4}\left[3 + \underbrace{e^{j\pi n}}_{(-1)^n}\right] = \left\{\underset{n=0}{1}, \overset{n=1}{\frac{1}{2}}, 1, \underset{n=3}{\frac{1}{2}}\right\}$$

$$x_2[n] = \frac{1}{4}\sum_{k=0}^{3} jX_I[k]e^{j\frac{2\pi}{4}nk} =$$

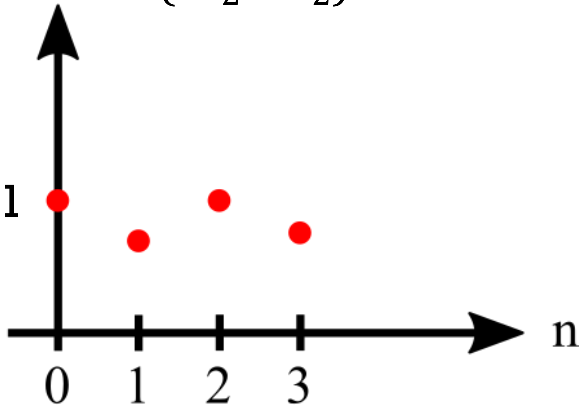
$$= \frac{1}{4}\left[-je^{j\frac{2\pi}{4}n} + je^{j\frac{2\pi}{4}n\cdot 3}\right] = \frac{1}{4}\left[-je^{j\frac{\pi}{2}n} + je^{j\frac{3\pi}{2}n}\right] = 0$$

We now insert $n$ and summarize:

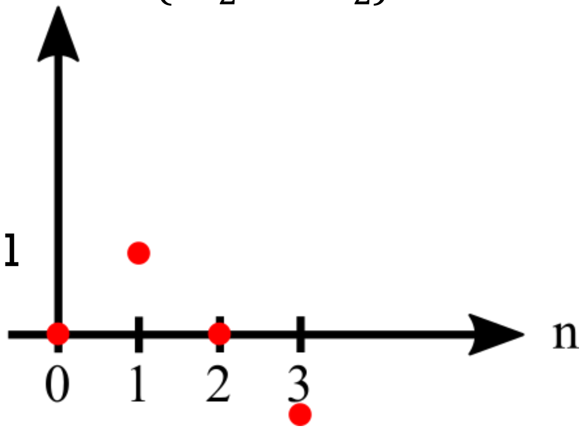
$$= \begin{cases} 0, & n = 0 \\ \frac{1}{4}[(-j)\cdot(j) + (j)\cdot(-j)] = \frac{1}{2}, & n = 1 \\ \frac{1}{4}[(-j)\cdot(-1) + (j)\cdot(-1)] = 0, & n = 2 \\ \frac{1}{4}[(-j)\cdot(-j) + (j)\cdot(j)] = -\frac{1}{2}, & n = 3 \end{cases}$$

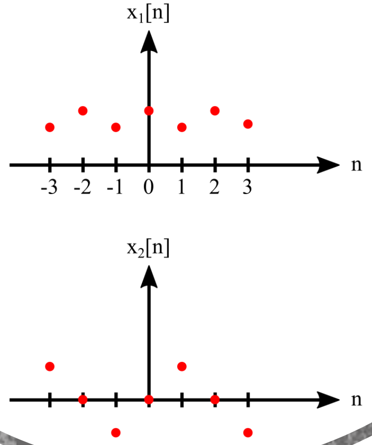
H.W.

$x_1[n] = \left\{1, \frac{1}{2}, 1, \frac{1}{2}\right\}$



$x_2[n] = \left\{0, \frac{1}{2}, 0, -\frac{1}{2}\right\}$



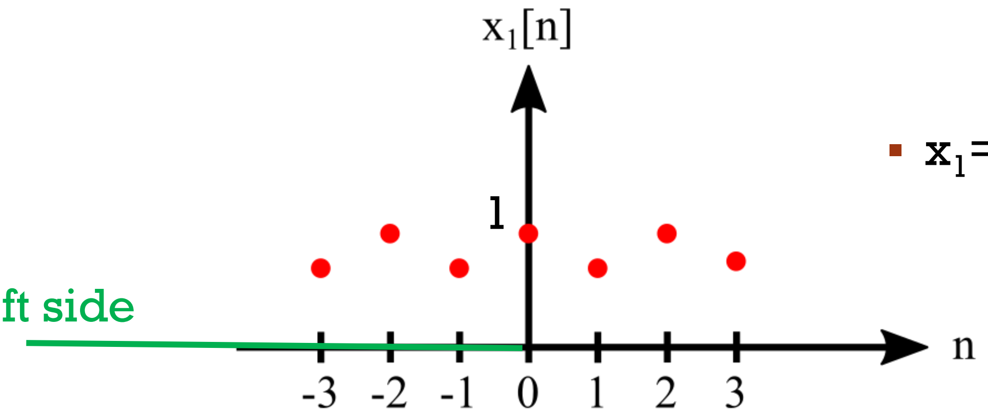


# VALIDATION

- $x_1 + x_2 = x$

Even or Odd is a property measured in FT as compared to 0

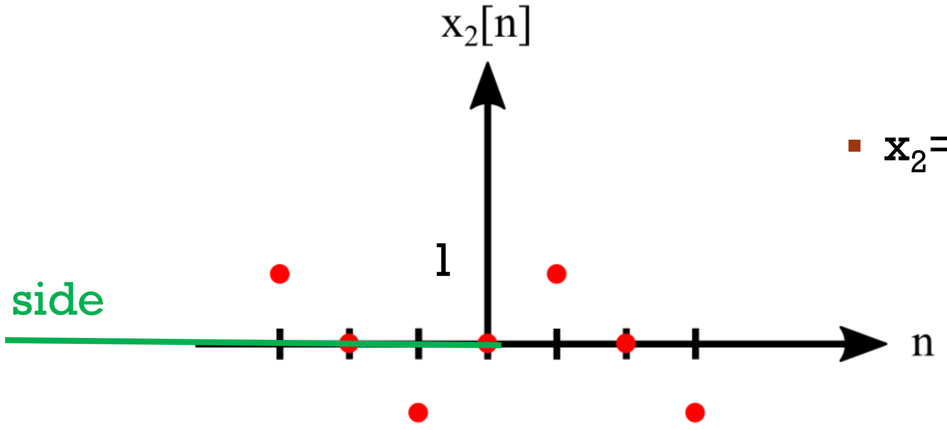
# SIMILAR PROPERTY: FT VS. DFT



Expanding the left side

- $x_1=x_e$ even in periodic sense

  periodic



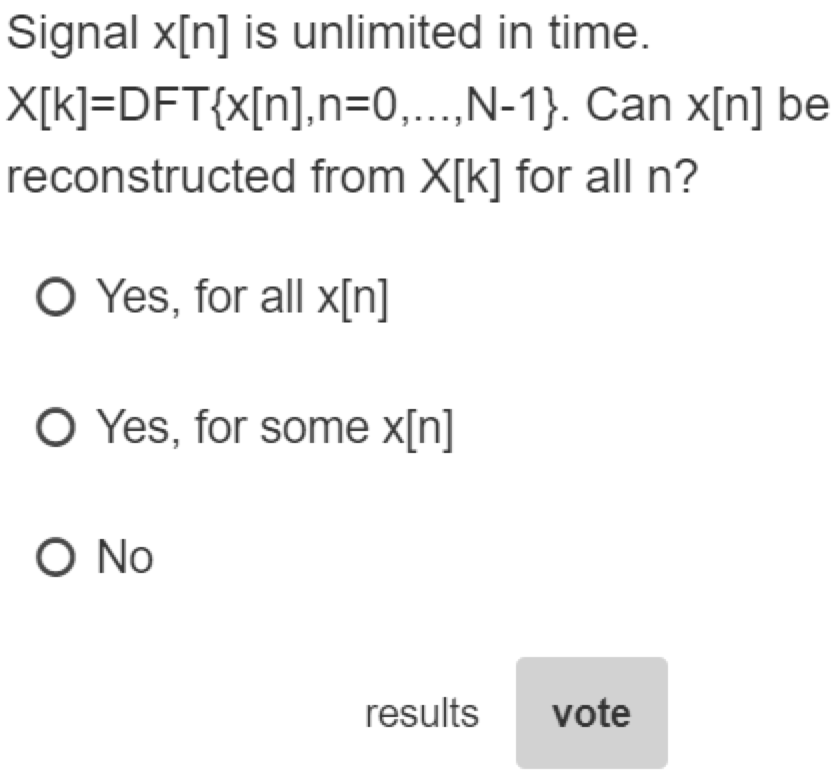
Expanding the left side

- $x_2=x_o$ odd in periodic sense

  periodic

# SURVEY: DFT TRANSFORM

- EasyPolls:



Signal x[n] is unlimited in time. X[k]=DFT{x[n],n=0,...,N-1}. Can x[n] be reconstructed from X[k] for all n?

○ Yes, for all x[n]

○ Yes, for some x[n]

○ No

תשובה:
אם זרקנו מידע
שאנחנו לא יודעים
אזי לא ניתן לשחזרו
– לכן תשובה ראשונה
לא נכונה

אם זרקנו מידע
שאנחנו יודעים אזי
ניתן לשחזרו חלקית

# DFT OF LONG SEQUENCES

What is the meaning of DFT when the sequence is longer than *N*?

# DFT OF LONG SEQUENCES

- Let assume that $x[n]$ is long (length $L > N$). We want to calculate DFT of length N, so we will sample in frequency at N points.

Then, there are two options:

Case 1: $x[n]$ is long and periodic

Case 2: $x[n]$ is long and non periodic

# DFT OF LONG SEQUENCES: CASE I

Case 1: $x[n]$ is long and periodic:

Assuming periodic signal $\tilde{x}[n]$ with period $N$ while $x[n]$ is one period of $\tilde{x}[n]$.

$\text{DFT}\{x[n]\} = X[k]$.

Calculate IDFT → Solution:

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} = x[n] \qquad 0 \le n \le N-1$$

Note: $e^{j\frac{2\pi}{N}kn}$ is periodic with period $N$ in $n$, therefore $= \tilde{x}[n] \quad n \in \mathbb{Z}$

From here, pair DFT $x[n], X[k]$ are periodic sums with period $N$ - this relation is named DTFS (**D**iscrete-**T**ime **F**ourier **S**eries).

# THE DFT AND DTFS

**Discrete-Time Fourier Series** **טור פורייה לסדרה**

- They are almost the same computation
- Different only in the normalization

**DTFS**

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

**DFT**

$$x[n] = \frac{1}{N}\sum_{r=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

# WHY ARE THE NORMALIZATIONS DIFFERENT?

- The DFT is a transform
  - It is sampled from the Fourier of a non-periodic function
  - It calculates the integral over $x$
  - The DFT will obey Parseval's theorem

$$x[n] = T_s x(nT_s)$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} T_s x(nT_s) e^{-jnk\omega_0 T} \approx \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

# DTFS VS. CTFS

- The DTFS is a series
  - The underlying function is periodic
  - The weight of a single coefficient is calculated as an average over one period
  - Parseval: sum of all the coefficients = average over time

**Discrete-Time Fourier Series**

**DTFS**

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

**Continuous-Time Fourier Series** טור פורייה

**CTFS=FS(Fourier Series)**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$$

# EXAMPLE OF DTFS

- We want to find the values of the coefficients
- They turn out to disappear for all values except the base frequency
- Note the coefficients are
    - Symmetric in amplitude
    - Anti-symmetric in phase

$$x[n] = \sum_{r=-10}^{9} D_r e^{j0.1\pi rn}$$

$$|D_1| = |D_{-1}| = \frac{1}{2}$$

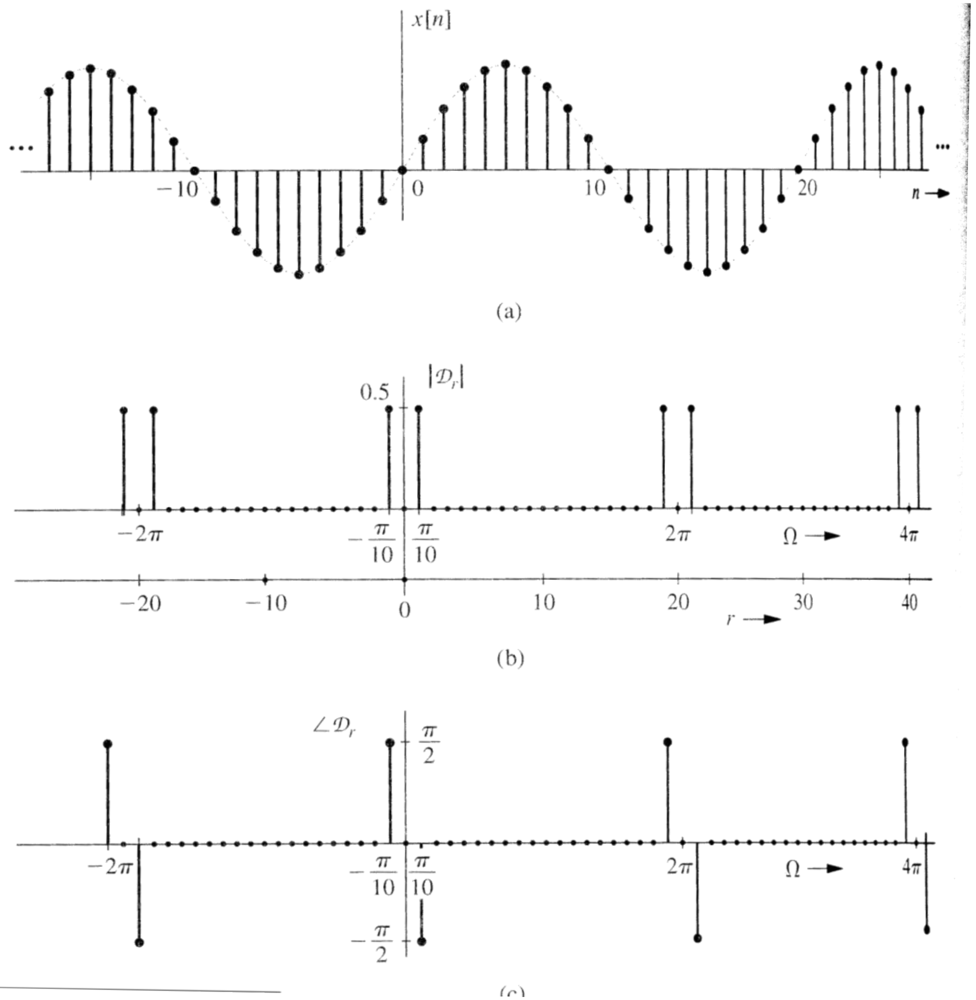
$$D_1 = \frac{1}{2j} = \frac{1}{2} e^{-j\frac{\pi}{2}}$$

$$\angle D_1 = -\frac{\pi}{2}; \quad \angle D_{-1} = \frac{\pi}{2}$$

$$D_{-1} = -\frac{1}{2j} = \frac{1}{2} e^{j\frac{\pi}{2}}$$

# FREQUENCY SPECTRUM FOR THE EXAMPLE

- The sine is a single frequency
- It is composed of two exponentials
  - One positive frequency
  - One negative frequency
- The spectrum is periodic and discrete



# DFT OF LONG SEQUENCES: CASE II

We assume that $x[n]$ is long of length $L > N$.

Case 2: $x[n]$ is long and non periodic:

- In this case, we can calculate DFT only on elements $0 \le n \le N-1$
- And so, based on the definition we will calculate the N points DTFT:

$$\hat{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[e^{j\overset{\theta_k}{\frac{2\pi}{N}k}}]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\underbrace{\sum_{m=-\infty}^{\infty} x[m]e^{-j\underset{\theta_k}{\frac{2\pi}{N}}km}}_{} e^{j\frac{2\pi}{N}kn} =$$

Sampled DTFT

$$= \sum_{m=-\infty}^{\infty} x[m]\underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k[n-m]}}_{} = \sum_{m=-\infty}^{\infty} x[m]\underbrace{\delta[(n-m)\mathrm{mod}\,N]}_{}$$

Get rid of modulo via $\delta$

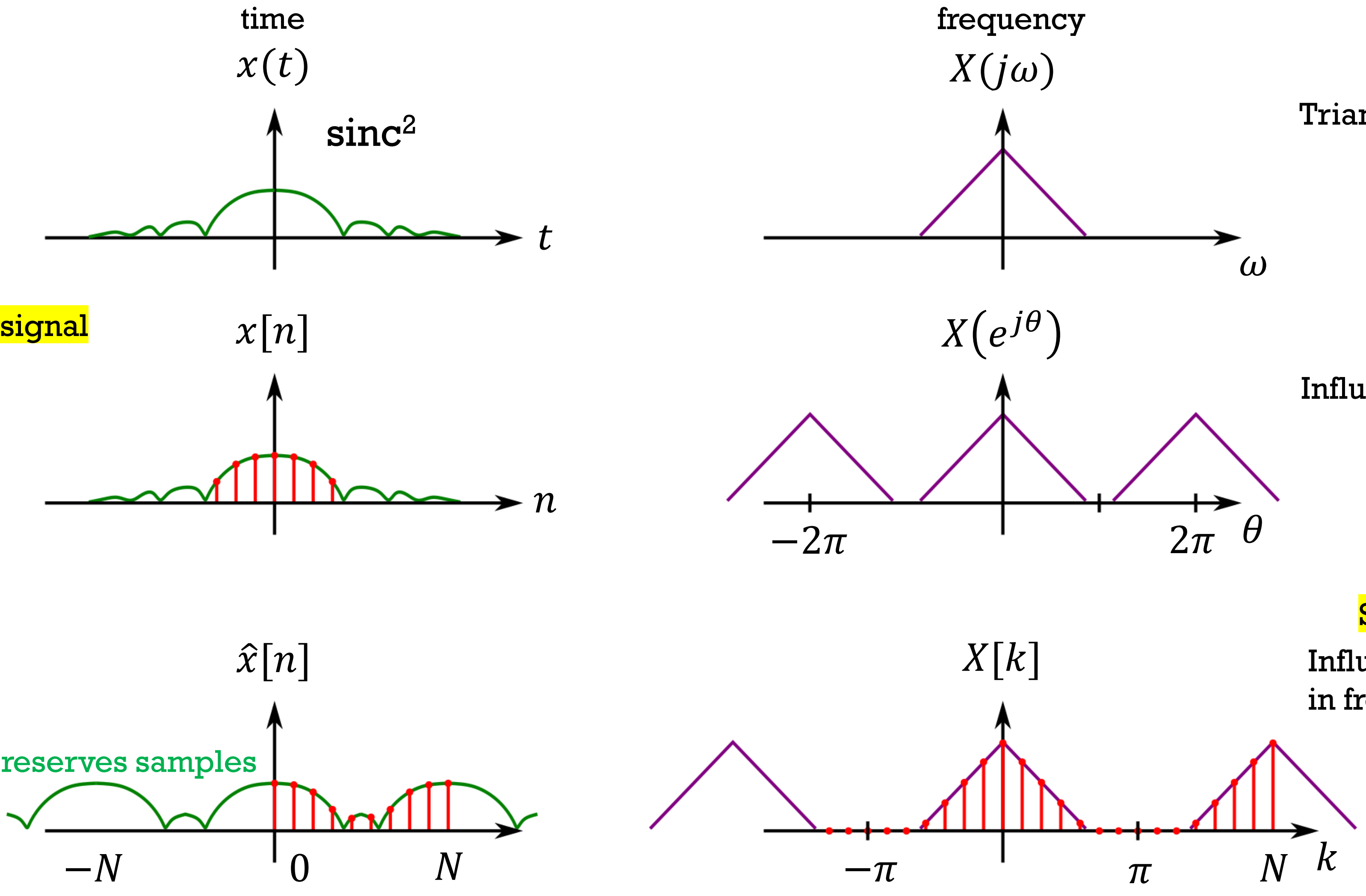
Impulse train

$$= \sum_{m=-\infty}^{\infty} x[m]\sum_{l=-\infty}^{\infty}\delta[n-m+lN] = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m]\delta[n+lN-m] =$$

$$= \sum_{l=-\infty}^{\infty} x[n+lN]$$ **sifting**

This reminds replications – sampling in frequency-> replication in time when $x[n]$ of length $N$, the replications will not overlap

# EXAMPLE: GRAPHICAL DESCRIPTION



Triangular window
FT

Sampled signal

Influence of sampling in time
DTFT

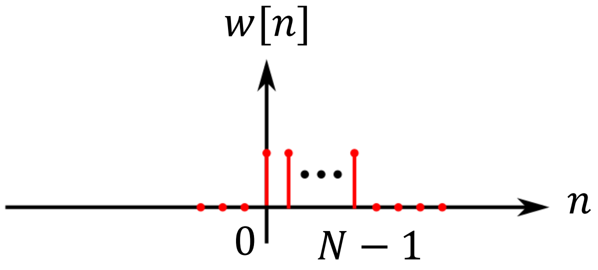
Sampled frequency

Influence of N-points sampling in frequency
DFT

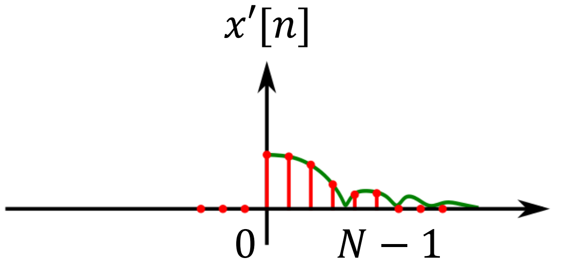
Periodic,
Replications
*this option preserves samples in frequency

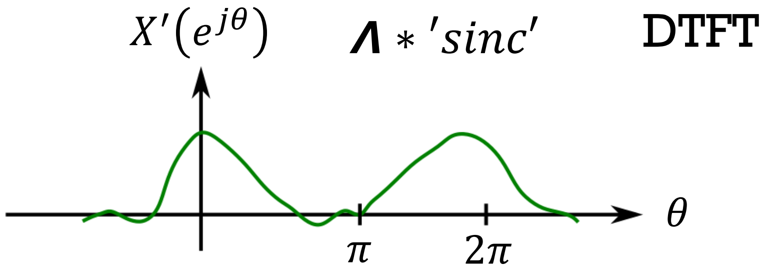
# EXAMPLE: GRAPHICAL DESCRIPTION – WE MULTIPLY X[N] BY A WINDOW

- In this case the replications in time will overlap.
- To avoid the replications, we will a apply on $x[n]$ window in time

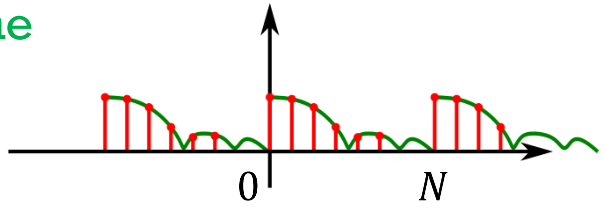


$$x'[n] = x[n] \cdot w[n]$$

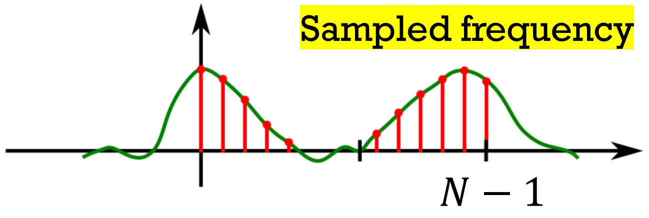




This option preserves the (limited) information in time

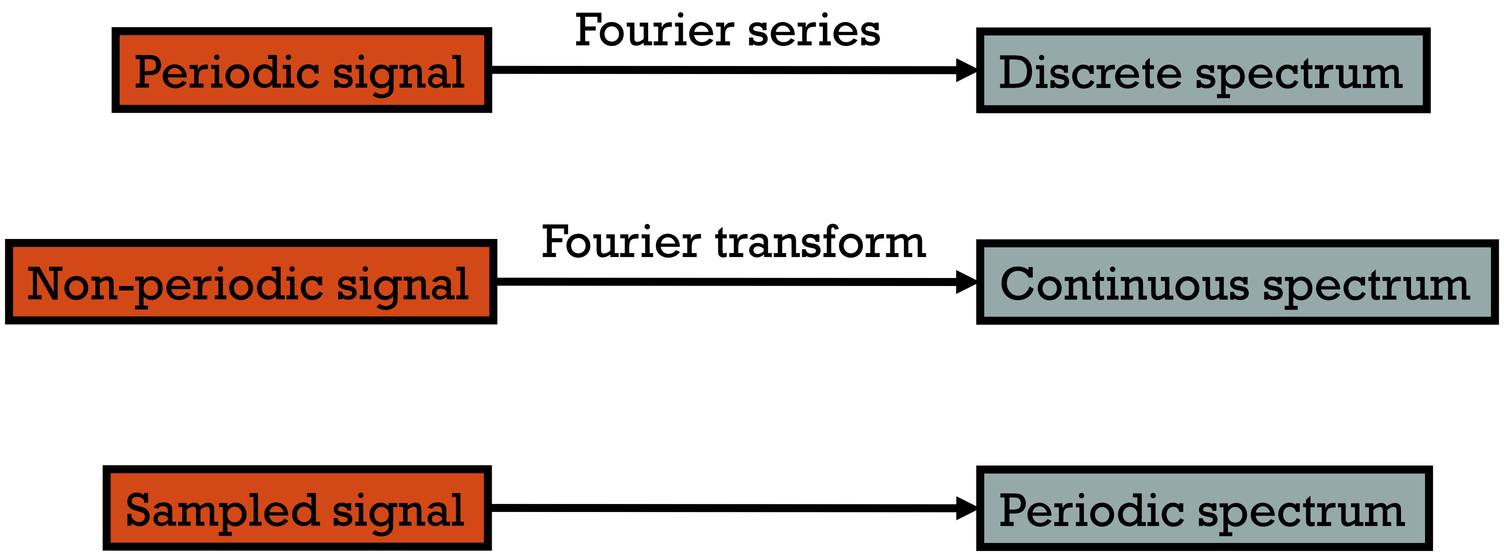


DFT



# FOURIER TRANSFORM RELATIONS

- There is a relationship between periodicity of a function and the continuous nature of its transform.



Periodic signal
Fourier series
Discrete spectrum

Non-periodic signal
Fourier transform
Continuous spectrum

Sampled signal
Periodic spectrum

# FOURIER TRANSFORM AND FOURIER SERIES

- By duality, we get more relationships

Periodic signal → Discrete spectrum

Discrete signal → Periodic spectrum

Non-periodic signal → Continuous spectrum

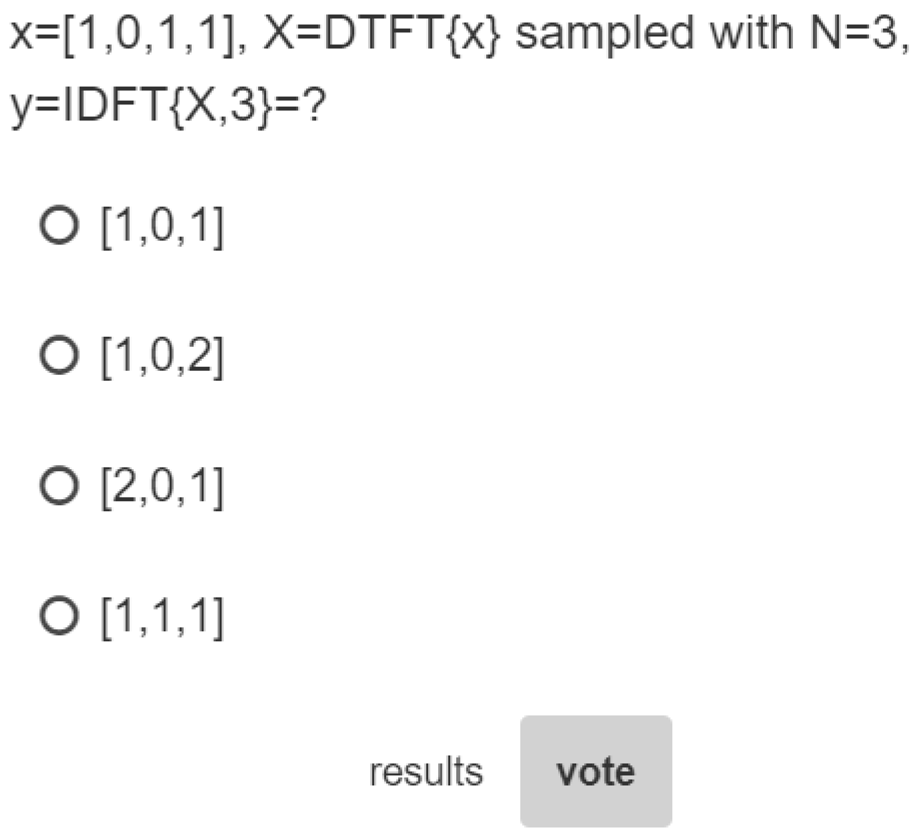
Continuous signal → Non-periodic spectrum

Sampled signal → Periodic spectrum

Periodic signal → Sampled spectrum

# SURVEY: SAMPLING IN FREQUENCY

- **EasyPolls:**



x=[1,0,1,1], X=DTFT{x} sampled with N=3, y=IDFT{X,3}=?

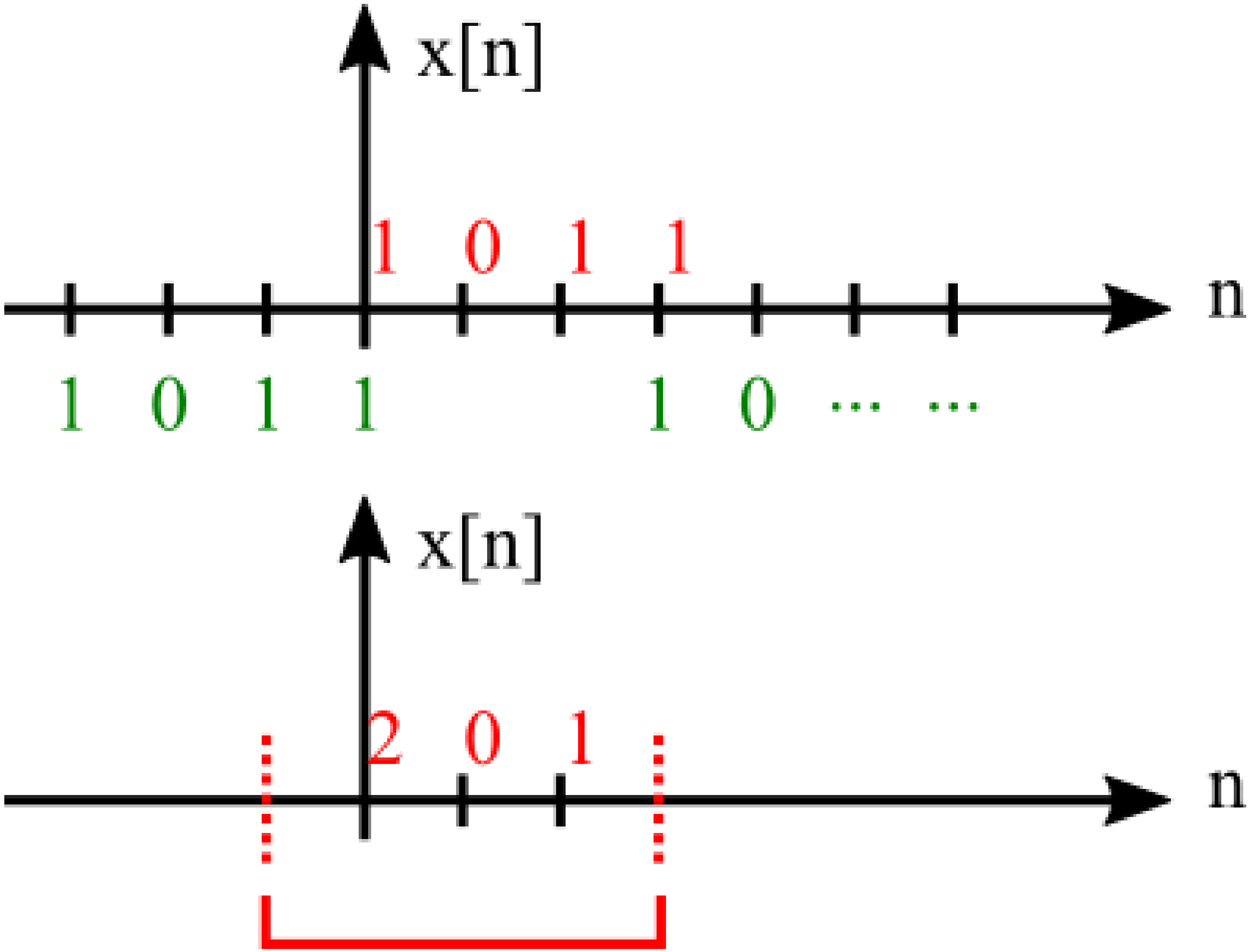
○ [1,0,1]

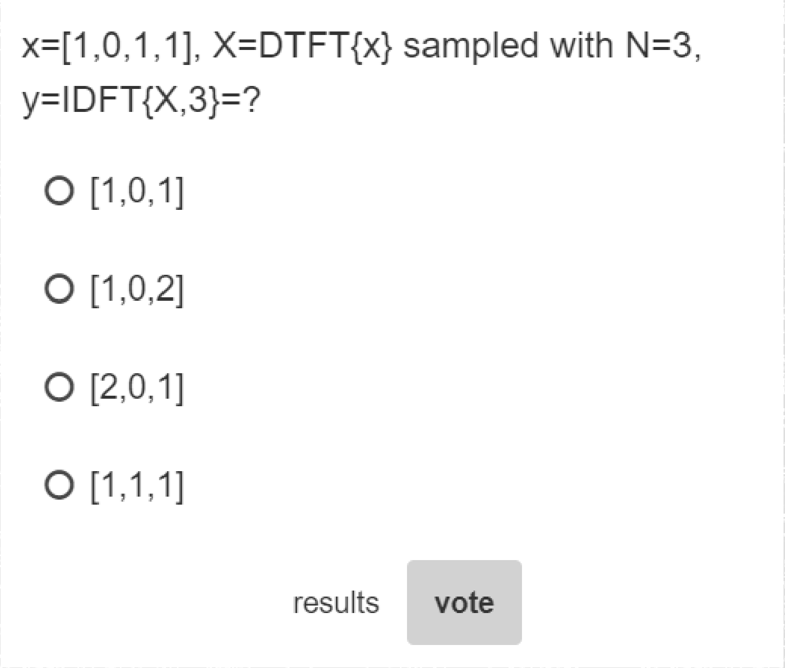
○ [1,0,2]

○ [2,0,1]

○ [1,1,1]

# SURVEY: SOLUTION




x=[1,0,1,1], X=DTFT{x} sampled with N=3, y=IDFT{X,3}=?

- ○ [1,0,1]
- ○ [1,0,2]
- ○ [2,0,1]
- ○ [1,1,1]

# DFT MATRIX REPRESENTATION

Notation via linear algebra

# MATRIX REPRESENTATION

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] W_N{}^{-kn}, \;\; 0 \le k \le N-1$$

- Matrix representation :ייצוג מטריציוני

- אם:

$$x[n] = \left[x[0], x[1], x[2], \ldots, x[N-1]\right]; \quad X[k] = \left[X[0], X[1], X[2], \ldots, X[N-1]\right]$$

נגדיר מטריצה $\mathrm{W_N}$:

$$\mathrm{W_N} = \begin{array}{cccccc}
n=0 & n=1 & n=2 & \cdots & n=N-1 & \\
W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 & k=0 \\
W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} & k=1 \\
W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} & k=2 \\
\vdots & & & \cdots & & \vdots \\
W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} & k=N-1
\end{array}$$

נוח לייצג התמרת DFT כהכפלה של אות הרצוי במטריצה $W_N$:

$$W_N = e^{j\frac{2\pi}{N}}$$

$$X_N[k] = W_N \cdot x_N[n]$$

$$\underline{X} = \underline{\underline{F}}\,\underline{x}$$

$\underline{\underline{F}}$ is named ‘DFT matrix’

Note: it reminds F matrix in non-uniform sampling

# DFT IN MATRIX REPRESENTATION

$$N\text{x}N$$

$$N\text{x}1 \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \times \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} N\text{x}1$$

# MATRIX REPRESENTATION: EXAMPLES FOR N=2, 4

$$\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$$W_N = e^{j\frac{2\pi}{N}}$$

נתון N=2 : הצג/י בצורה מטריציונית.

פתרון : $X_2[k] = W_2 \cdot x_2[n]$

$$\underline{F} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

DC חיבור דגימות

AC דגימות חיסור

***D****irect* **C**urrent

***A****lternating* **C**urrent

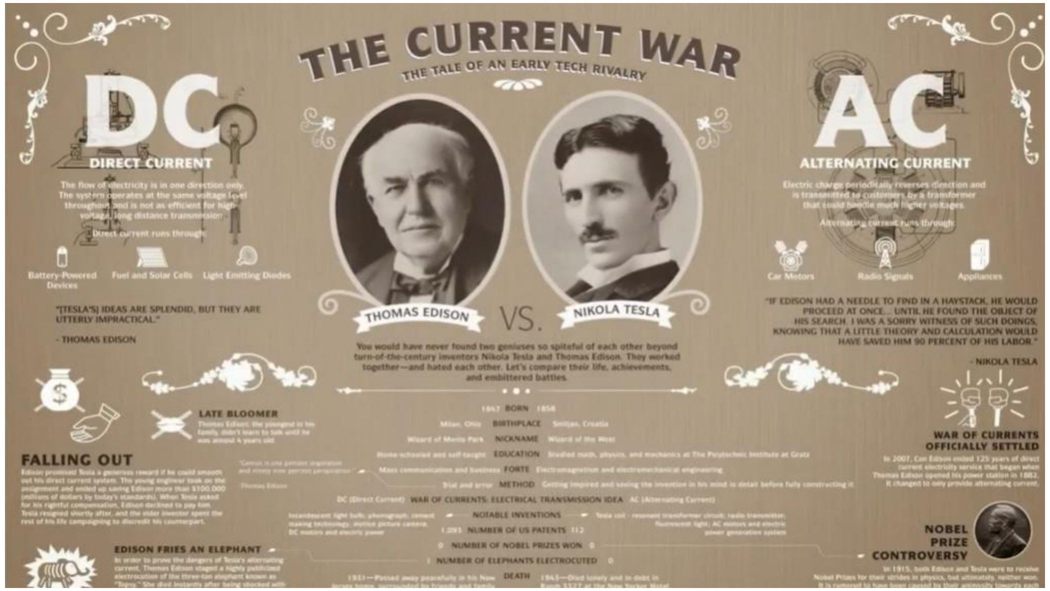
כאשר $W_N = e^{j\frac{2\pi}{N}}$ ואז

נתון N=4 : הצג/י בצורה מטריציונית.

פתרון : $X_4[k] = W_4 \cdot x_4[n]$

$$\underline{F} = \begin{array}{c} {}_{k}\diagdown^{n} \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_{\mathrm{N}}^{-kn} = e^{-j\frac{2\pi}{4}nk} = e^{-j\frac{\pi}{2}nk}$$



# WAR OF CURRENTS

- In 1912, the Nobel Committee announced that Nikola Tesla and Thomas Edison were the recipients of the Physics Prize; instead, the prize went to Gustav Dalen. Details of the reversal are unclear but it is known that **Tesla refused the prize** (and the $20,000 that came with it).

# MATRIX REPRESENTATION: EXAMPLE N=3

H.W.

נתון N=3 : הצג/י בצורה מתרצית.

פתרון : $X_3[k] = W_3 \cdot x_3[n]$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_N^{-2} \\ 1 & W_3^{-2} & W_N^{-4} \end{bmatrix}$$

כאשר $W_N = e^{j\frac{2\pi}{N}}$ ואז

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-2\cdot j\frac{2\pi}{3}} \\ 1 & e^{-2\cdot j\frac{2\pi}{3}} & e^{-4\cdot j\frac{2\pi}{3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & cos\left(\frac{2\pi}{3}\right) - j\,sin\left(\frac{2\pi}{3}\right) & cos\left(2\cdot\frac{2\pi}{3}\right) - j\,sin\left(2\cdot\frac{2\pi}{3}\right) \\ 1 & cos\left(2\cdot\frac{2\pi}{3}\right) - j\,sin\left(2\cdot\frac{2\pi}{3}\right) & cos\left(4\cdot\frac{2\pi}{3}\right) - j\,sin\left(4\cdot\frac{2\pi}{3}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

# MATRIX REPRESENTATION: EXAMPLE

In linear algebra, a complex square matrix *U* is unitary if its conjugate transpose *U** is also its inverse, that is, if

$$U^*U = UU^* = UU^{-1} = I,$$

Identity matrix $I_1 = \begin{bmatrix}1\end{bmatrix},\ I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ \ldots,\ I_n = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}.$

In physics, especially in quantum mechanics, the conjugate transpose is referred to as the Hermitian adjoint of a matrix and is denoted by a dagger (†), so the equation above is written $U^\dagger U = UU^\dagger = I.$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2}\\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2}\end{bmatrix}$$

נשים לב כי:

1) $W_N^{kn} = W_N^{nk}$

2) $\frac{1}{\sqrt{N}} W_N$ - מטריצה יוניטרית.

הגדרה – מטריצה יוניטרית: $W_N \cdot \bar{W}_N' = N I_N$

הערה:

$\bar{W}$ הינו צמוד קומפלקסי של $W$

$W'$ הינה מטריצה הפכית של $W$

# DEMONSTRATION OF DFT

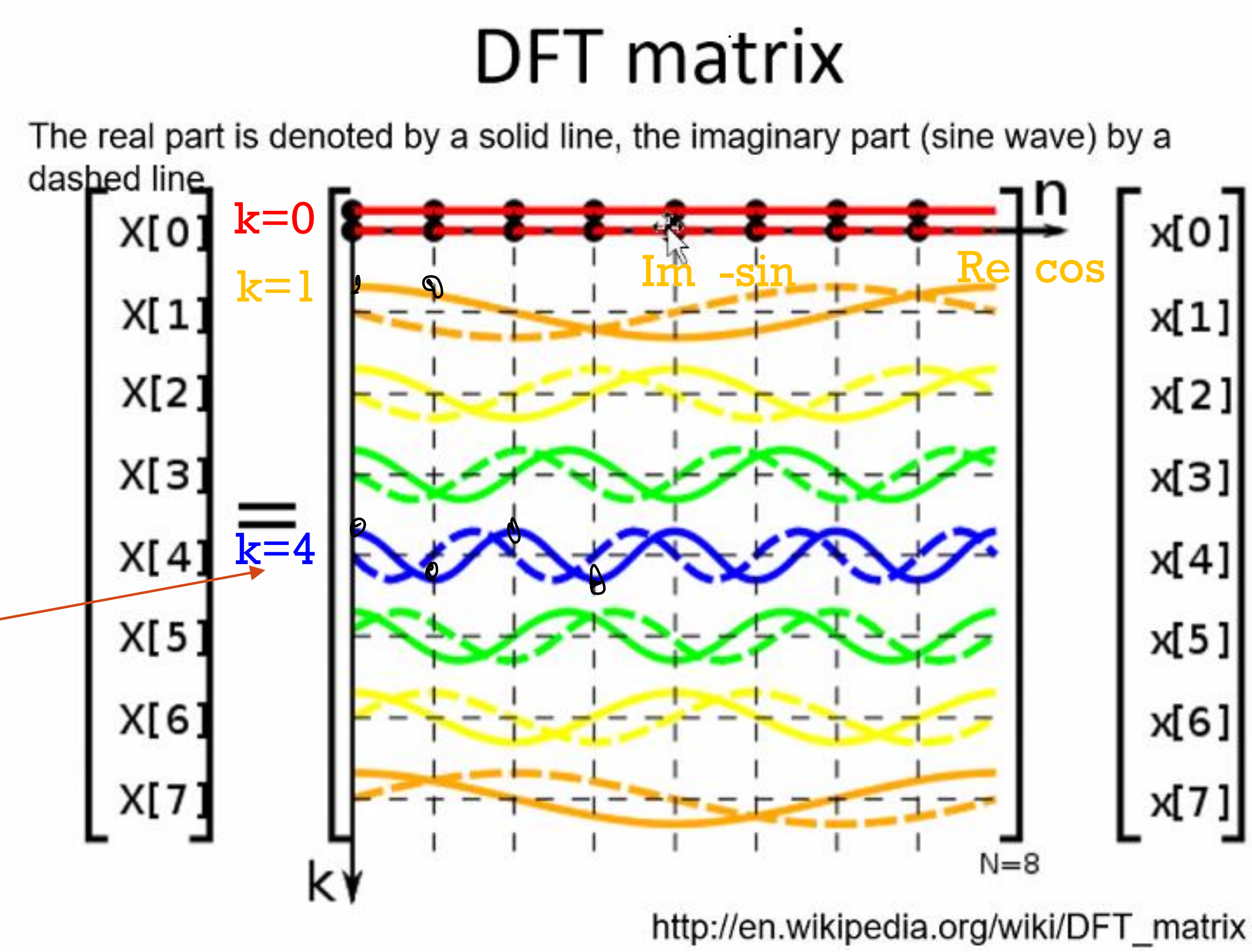
Exponents here are presented graphically

N=8

positive k

highest change: change in each sample

negative k



DFT matrix

The real part is denoted by a solid line, the imaginary part (sine wave) by a dashed line

http://en.wikipedia.org/wiki/DFT_matrix

# IDFT: MATRIX REPRESENTATION

From IDFT definition: $\underline{x} \quad = \frac{1}{N}\underline{F}^{*}\underline{X}$

Inverse matrix DFT $\quad = \underline{F}^{-1}\underline{X}$

$$\sum_{k=0}^{N-1} W_N^{kn} W_N^{-km} = N \cdot \delta[n-m]$$

Unitary matrix (orthonormal and complex)

$$\left(\frac{1}{\sqrt{N}}\underline{F} \quad \right)^{H}\left(\frac{1}{\sqrt{N}}\underline{F}\right) = \underline{I}$$

$$(F_N)_{k,m} = W_{\mathrm{N}}^{-\mathrm{km}}$$

*In Matlab: X′ is both Hermite and transpose*

but X.’ is just transpose