DISCRETE FOURIER TRANSFORM (DFT)

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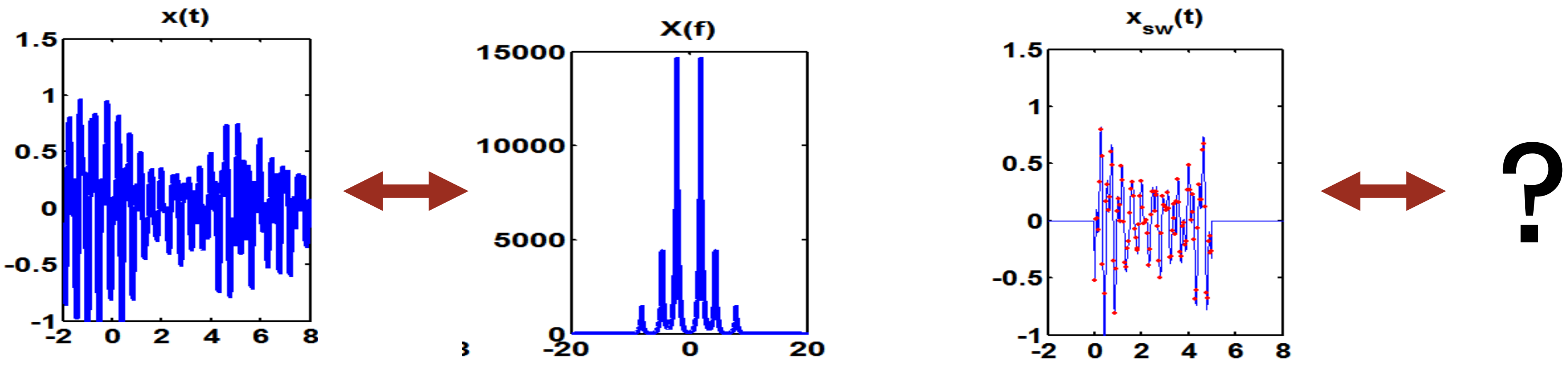
Reading: Chapter 4 by B. Porat

DISCRETE FOURIER TRANSFORM (DFT)

- In real life situations:
  - We have sampled a signal
  - We want to use a computer to analyze the signal in Fourier domain - > Fourier transform
- In this case:
  - The signal is time limited
  - The signal is discrete

THE ORIGINAL SIGNAL AND THE SAMPLED SIGNAL

- We want the spectrum of the real signal
- We only have the sampled, time-limited signal



THE FT DILEMMA

- The actual signal is not really time limited
- The actual signal is continuous
- The Fourier transform is continuous
    - But we can only sample it at discrete frequencies
- The Fourier transform is not necessarily band-limited
    - But we can only sample at a finite number of frequencies

THE FT SITUATION IN THIS CASE

- We sampled a function for a limited time
- We want to estimate its spectrum
- Our estimate of the spectrum will be sampled and for limited bandwidth

THE DFT SOLUTION

1) Assume, the time limited signal repeats outside of the window
    - This will give us a discrete Fourier transform (DFT)

2) Assume, the Fourier transform repeats outside of the window (DTFT)
    - This will give us a sampled signal in time

- When we combine 1) + 2) to obtain:
    - Time limited, sampled signal
    - Band limited, sampled Fourier transform

DFT

- Start with a signal

$$x(t) \leftrightarrow X(j\omega)$$

- Look at it for a finite time, T_0

$$x_f(t) = x(t)\Pi\left(\frac{t - \frac{T_0}{2}}{T_0}\right) \qquad \leftrightarrow X(j\omega) = X(j\omega) * \mathcal{F}\left[\Pi\left(\frac{t - \frac{T_0}{2}}{T_0}\right)\right]$$

- Sample at a rate of T_s

$$x_{sf}(t) = x_f(t)\sum_{n=0}^{\frac{T_0}{T_s}} \delta(t - nT_s) \qquad \leftrightarrow X_{sf}(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k2\pi/T_s)$$

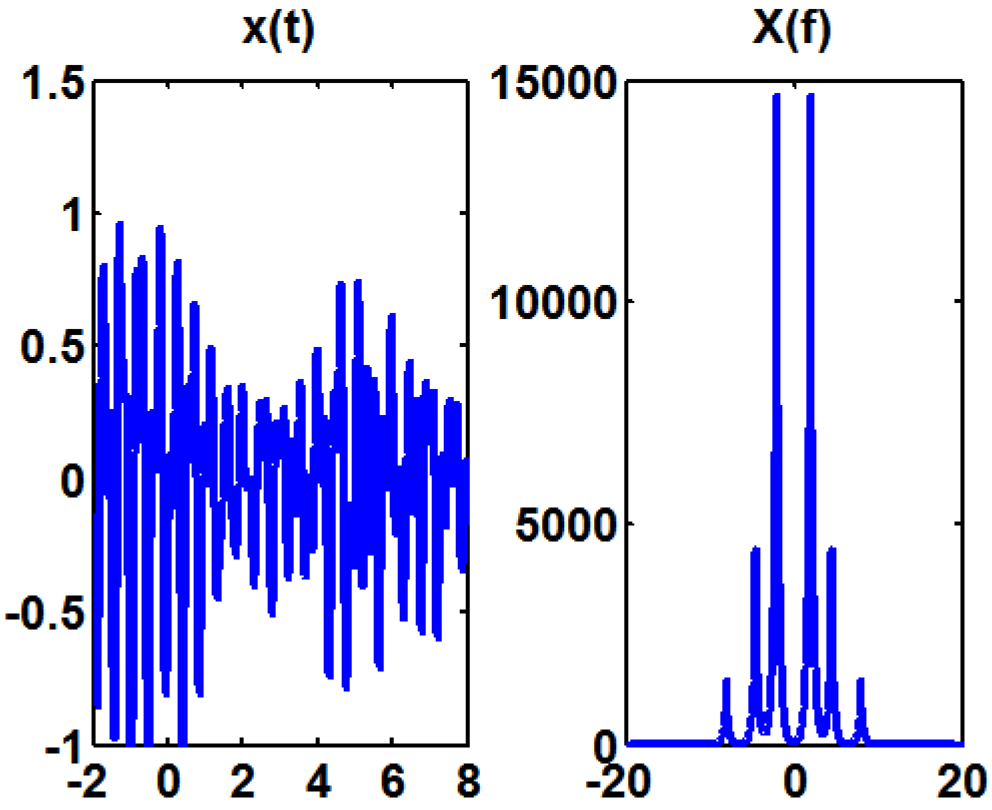
DFT

- Sample the Fourier at a rate of $1/T_0$

$$x_{ssf}(t) = T_0 \sum_{n=-\infty}^{\infty} x_{sf}(t - nT_0) \leftrightarrow \qquad X_{ssf}(j\omega) = X_{sf}(j\omega) \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T_0}\right)$$

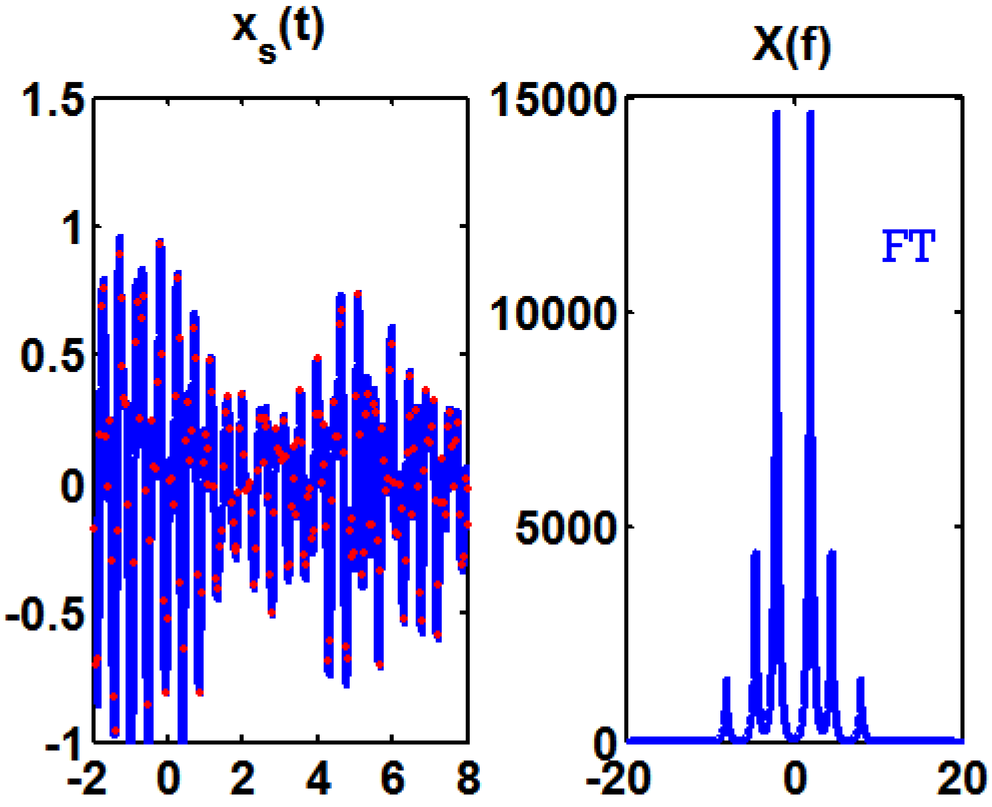
MATLAB EXAMPLE

- A signal and its Fourier transform



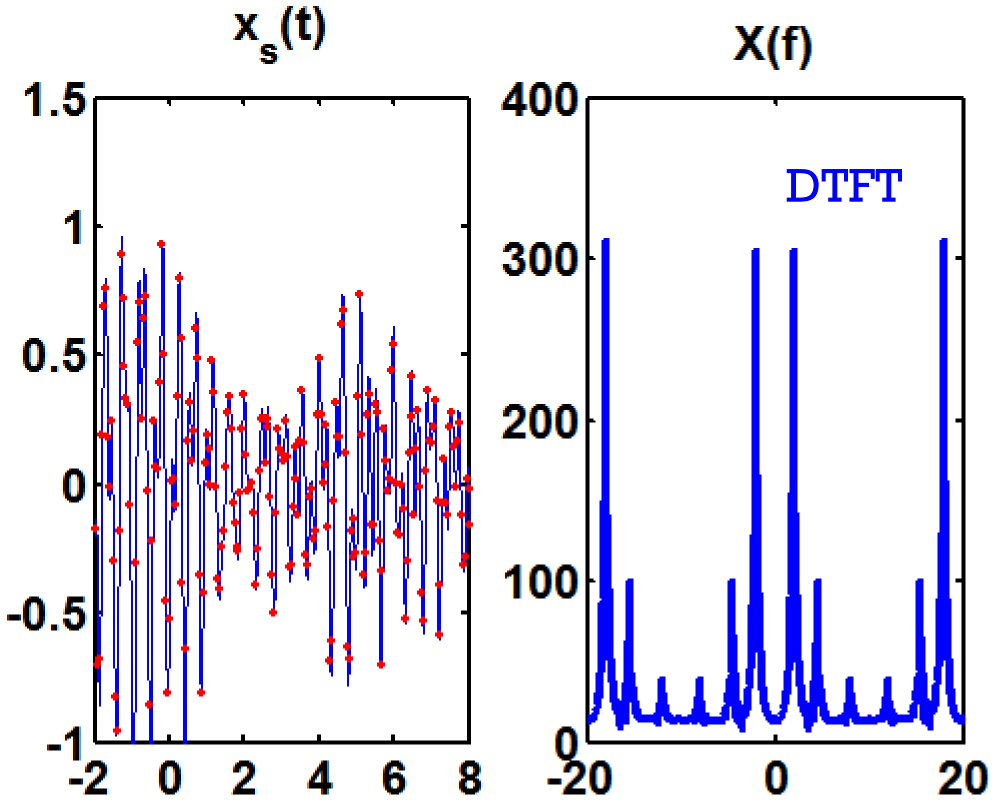
MATLAB EXAMPLE

- First, we'll sample it



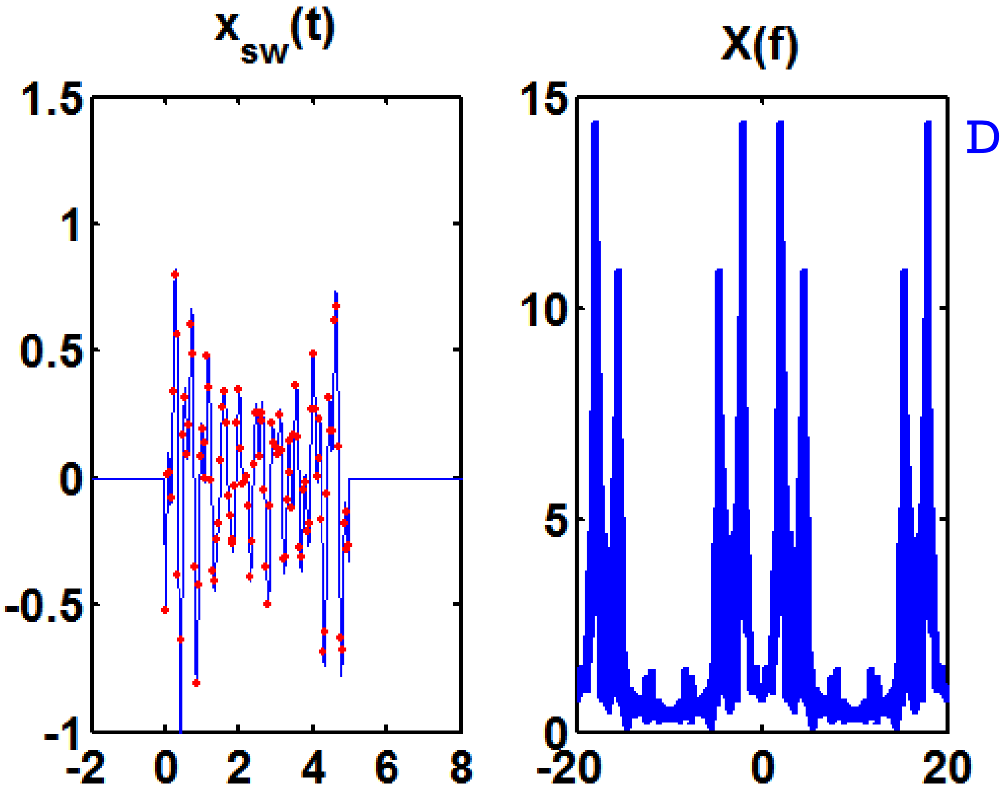
MATLAB EXAMPLE

- That causes replication of the Fourier spectrum
  - This can cause aliasing
- Note also change in $X(f)$ scale



MATLAB EXAMPLE

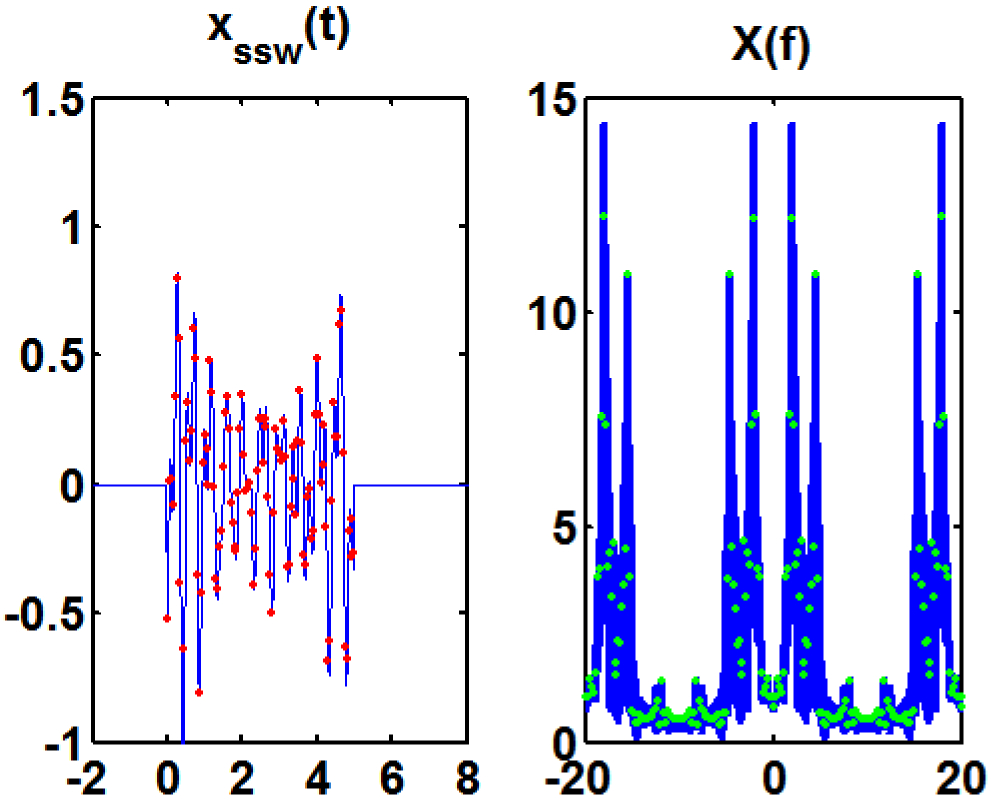
- Now we window in time because we can only observe for finite time
- Windowing with a rectangular window causes convolution with a *sinc* function
  - This is called leakage



DTFT*sinc

MATLAB EXAMPLE

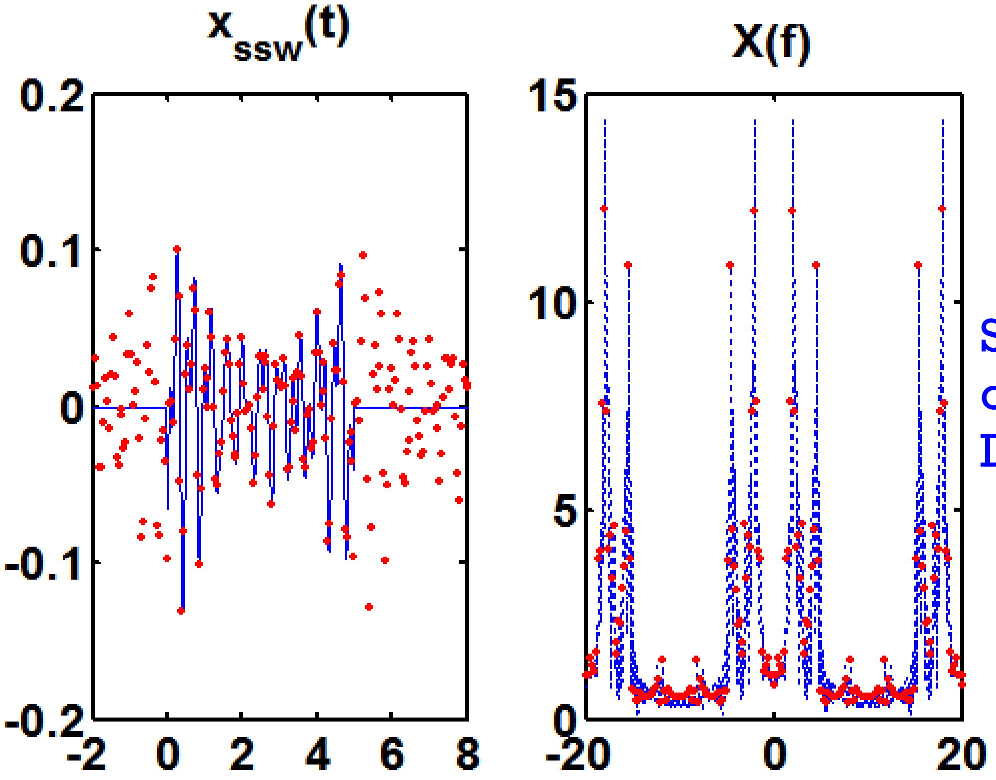
- Now we sample the frequency spectrum



DTFT*sic

MATLAB EXAMPLE

- Now we sample the frequency spectrum
- Sampling the frequency spectrum turns the time domain into a periodic function



Sampling of DTFT*sic

DFT

- Start with a signal
- Look at it for a finite time, T_0
- Sample at a rate of T_s
- Sample the Fourier at a rate of $1/T_0$

- We end up with two sampled periodic functions
  - A periodic, sampled function of time
    - Period T_0, sample size T_s
  - A periodic, sampled function of frequency
    - The two functions are transforms of each other
    - The function of frequency approximates the transform of the original signal

DISCRETE TIME FOURIER TRANSFORM

- The discrete time Fourier transform (DTFT), $X(e^{j\theta})$, of a discrete time function $x[n]$ of length $\boldsymbol{N}$, is a continuous spectrum of frequencies
    - This is because there is no lower limit on the frequency of a non-periodic function
- The integral is over a range of 2π frequencies
    - This is because there is an upper limit on the frequency in the discrete signal

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta$$

$$\theta \in [-\pi, \pi]$$

$$X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\theta n}$$

DTFT

CONJUGATE SYMMETRY OF FOURIER

- For **real functions**, this gives us:
  - Even symmetry for the magnitude
  - Odd symmetry for the phase

$$\mathrm{DTFT}\{x^*[n]\} = X^*\left(-e^{j\theta}\right)$$

So, if $x[n]$ is real:

$$X\left(e^{j\theta}\right) = \mathrm{DTFT}\{x[n]\} = \mathrm{DTFT}\{x^*[n]\} = X^*\left(-e^{j\theta}\right)$$

$$X\left(e^{j\theta}\right) = X^*\left(-e^{j\theta}\right)$$

$$\left|X\left(e^{j\theta}\right)\right| = \left|X\left(-e^{j\theta}\right)\right| \qquad \angle X\left(e^{j\theta}\right) = -\angle X\left(-e^{j\theta}\right)$$

DTFT (DISCRETE TIME FOURIER TRANSFORM)

הגדרה: התמרת פורייה מחזורית ורציפה ל**<u>סדרה סופית</u>**

נניח אות דגום: $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$

נחשב התמרת פורייה של האות $x_s(t)$

$$\begin{aligned}
\Im\{x_s(t)\} &= \int_{-\infty}^{\infty} x_s(t)e^{-j\theta t}\,dt = \\
&= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right]e^{-j\theta t}\,dt = \\
&= \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT)\right]e^{-j\theta t}\,dt = \\
&= \sum_{n=-\infty}^{\infty} x(nT)\,e^{-j\theta nT}
\end{aligned}$$

- נחליף ωT [rad/sec] בω [rad] על מנת להעלים את התלות בזמן רציף.
- נחליף $x(nT)$ בסדרה $x[n]$ על מנת להעלים את התלות בזמן רציף

כך נקבל את התמרת ה **DTFT**

$$\text{DTFT } X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

- בכניסה להתמרה: <u>האות הוא בדיד</u> (סדרת מספרים)
- <u>ההתמרה</u> המתקבלת הינה <u>רציפה בתדר</u>.
- <u>ההתמרה</u> המתקבלת הינה <u>מחזורית</u>.

IDTFT (INVERSE DISCRETE TIME FOURIER TRANSFORM)

- כעת נפתח ביטוי ל IDTFT :

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, e^{j\theta n} d\theta = \frac{1}{2\pi}\int_{2\pi}\left[\sum_{m=-\infty}^{\infty} x[m] e^{-j\theta n}\right] e^{j\theta n} d\theta =$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot \frac{1}{2\pi}\int_{2\pi} e^{j\theta(n-m)}\, d\theta = \frac{1}{2\pi}\frac{e^{j\theta(n-m)}}{j(n-m)} |_0^{2\pi} =$$

$$= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

כלומר:

$$\text{IDTFT} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, e^{j\theta n} d\theta$$

DTFT CONVERGENCE

- התכנסות ה DTFT:
- תנאי ההתכנסות דומים לאלה של תנאי ההתכנסות של התמרת פורייה אך מכיוון ש $x[n]$ היא סידרה (אות בדיד) אזי תנאי אי-רציפות אינו רלוונטי בהתמרה זו ולכן, התכונות הן :

1. אינטגרביליות בהחלט (absolutely integrable)

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2. אנרגיה סופית (final energy)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

PROPERTIES OF DTFT

- Time and frequency reversal

$$\mathrm{DTFT}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]\, e^{-j\theta n} = \sum_{m=-\infty}^{\infty} x[m] e^{j\theta m} = X\left(e^{-j\theta}\right)$$

TIME-SHIFT AND FREQUENCY SHIFT

- Easily proved
    - And you've showed it before

$$x[n-k] \Leftrightarrow X(\theta)e^{-jk\theta}$$

$$x[n]e^{j\theta_c n} \Leftrightarrow X(\theta-\theta_c)$$

LINEAR CONVOLUTION

- Convolution in time produces multiplication in frequency

$$x_1[n] * x_2[n] \Leftrightarrow X_1\left(e^{j\theta}\right)X_2\left(e^{j\theta}\right)$$

CONVOLUTION

- Multiplication in time produces ***periodic convolution*** in frequency
    - This is an integral over one period of the function, allowing for the repetition of the periodic functions

$$x_1[n]x_2[n] \Leftrightarrow X_1\left(e^{j\theta}\right) \odot X_2\left(e^{j\theta}\right)$$

$$X_1\left(e^{j\theta}\right) \odot X_2\left(e^{j\theta}\right) = \frac{1}{2\pi} \int_{2\pi} X_1(u) X_2(\theta - u) du$$

PARSEVAL'S THEOREM

- Also just like continuous time and just like Laplace and Z
  - Transforms preserve the amount of energy in the signal

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X\left(e^{j\theta}\right)\right|^2 d\theta$$

DTFT AND Z TRANSFORM - REMINDER

- The DTFT is almost like the z transform evaluated on the unit circle
- We have to be careful here, like with the Laplace
    - Unit circle must fall into ROC

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

$$z = e^{j\theta}$$

$$X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{j\theta}\right)^{-n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n} = X(j\omega)$$

DISCRETE FOURIER TRANSFORM (DFT)

עד כה דיברנו על התמרה רציפה DTFT של סידרה $x[n]$:

משוואת אנליזה:

$$\textbf{DTFT} \quad X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

משוואת סינתזה:

$$\text{IDTFT} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, e^{j\theta n} d\theta$$

כעת נבחר תחום : $0 \le n \le N-1$, כאשר **N- אורך של סידרה בדידה**, ונקבל : $X(e^{j\theta}) = \sum_{n=0}^{N-1} x[n]e^{-j\theta n}$

כאשר $e^{j\theta}$ סימון להתמרת DTFT. יחידות של θ הן rad

אם נדגום את ציר התדר : $0 \le k \le N-1$, $\theta[k] = \frac{2\pi k}{N}$, נוכל לקבל את התמרת הפורייה הבדידה :

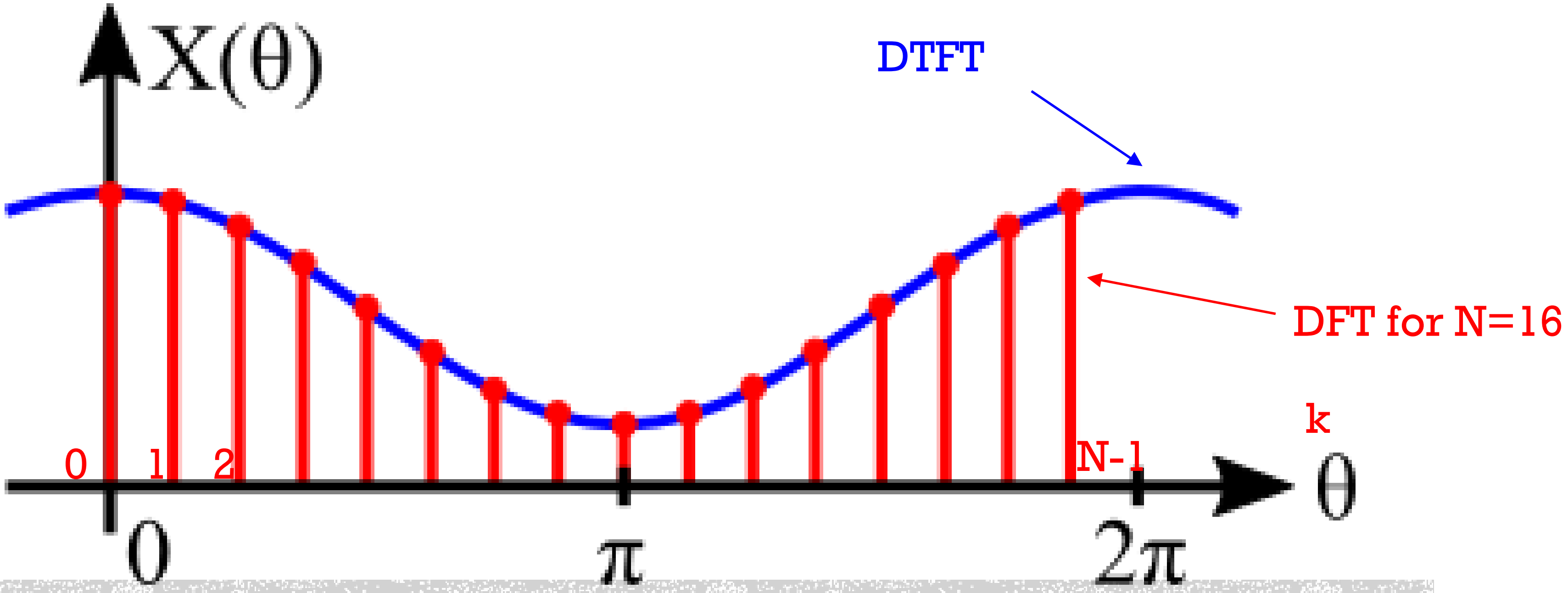
$$\mathrm{DFT}\{x[n]\} = X[k]$$

$$\textbf{DFT} \quad X[k] = X(e^{j\theta})\big|_{\theta=\frac{2\pi}{N}k} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

נגדיר: twiddle factor $W_N = e^{j\frac{2\pi}{N}}$ ואז

$$\textbf{DFT} \quad X[k] = \sum_{n=0}^{N-1} x[n]{W_N}^{-kn}, \quad 0 \le k \le N-1$$

$$X[k] = X\left(e^{j\theta}\right)\Big|_{\theta=\frac{2\pi}{N}k} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$



EXAMPLE: DISCRETE FOURIER TRANSFORM (DFT)

DEFINING THE DFT

- Start with
  - A function of time, $x(t)$, and its spectrum, $X(j\omega)$
  - A finite sampled time signal, $x(nT)$ → transform $X\left(e^{j\theta}\right)$
  - Its finite sampled spectrum, $X\left(e^{jn\theta}\right)$ with $\theta[k] = \frac{2\pi k}{N}$

Define:

$$x[n] = Tx(nT) = \frac{T_0}{N}x(nT)$$

$$X[k] = X(k\omega_0) = X\left(\frac{2\pi}{T_0}\right)$$

Then:

$$\theta = \omega_0 T = \frac{2\pi}{N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}$$

$x[n]$ and $X[k]$ are periodic sequences with period N.

PROVING THAT X[N] AND X[K] ARE TRANSFORMS

Write the sampled signal:

$$x_s(t) = \sum_{n=0}^{N-1} x(nT)\delta(t - nT)$$

Take its Fourier transform:

$$X_s(j\omega) = \sum_{n=0}^{N-1} x(nT)e^{-jn\omega T}$$

For $|\omega| \leq \omega_s/2$, we have just the first replication

$$X_s(j\omega) = \frac{X(j\omega)}{T}$$

So, we can write

$$X(j\omega) = T\sum_{n=0}^{N-1} x(nT)e^{-jn\omega T} \quad \text{for } |\omega| \leq \frac{\omega_s}{2}$$

PROVING THAT X[N] AND X[K] ARE TRANSFORMS

We have just written

$$X(\omega) = T\sum_{n=0}^{N-1} x(nT)e^{-jn\omega T} \quad \text{for } |\omega| \le \frac{\omega_s}{2}$$

So, that means

$$\mathrm{X}[k] = X(k\omega_0) = T\sum_{n=0}^{N-1} x(nT)e^{-jnk\omega_0 T}$$

$$= \sum_{n=0}^{N-1} Tx(nT)e^{-jnk\omega_0 T}$$

$$= \sum_{n=0}^{N-1} x_n e^{-jn\frac{2\pi}{N}k}$$

$$\Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

The proof for the other direction is just the same

ANOTHER WAY TO PROVE THE DFT RELATIONSHIP

- Now we're going the other direction
- Start with the DFT definition of $X[k]$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$$

- Multiply both sides by an exponent and sum over k

$$\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}km} = \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \right] e^{jk\frac{2\pi}{N}m}$$

- Now, if we switch the order of the summations, everything will fall out and we will end up with

$$\sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}m} = x[m]$$

INTRODUCING A NEW NOTATION FOR DFT

- The DFT as a sum of powers of roots of unity

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} x[n] \left(e^{-j\frac{2\pi}{N}}\right)^{kn} = \sum_{n=0}^{N-1} x[n] \left(e^{j\frac{2\pi}{N}}\right)^{-kn} = \sum_{n=0}^{N-1} x[n] (W_N)^{-kn}$$

0<=k<=N-1

IDFT $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] (W_N)^{kn}$$

$$\boxed{W_N = e^{j\frac{2\pi}{N}} = \sqrt[\frac{1}{N}]{1}}$$

DFT: PROPERTIES

- 1. $W_N{}^N = W_N{}^0$, הסבר:

$$W_N{}^N = e^{j\frac{2\pi}{N}N} = e^{j2\pi} = \cos(2\pi) + j\sin(2\pi) = 1 + 0 = 1 W_N{}^0 = e^{j\frac{2\pi}{N}0} = 1$$

וזאת משום שהסדרה $W_N{}^n$ מחזורית במחזור N

- 2. $\sum_{n=0}^{N-1} 1 \cdot W_N{}^{kn} = \begin{cases} N, & k \bmod N = 0 \\ 0, & \text{else} \end{cases} = N\delta[k \bmod N]$ $\qquad x[n] = 1$

אם k מתחלק ב $\boldsymbol{N}$ ללא שארית, סכום אשר לעיל שווה ל N, אחרת הוא שווה ל0.

הוכחה:

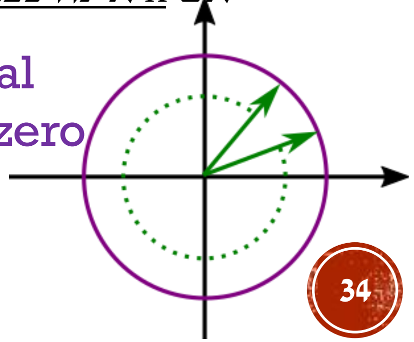
אם k <u>כפולה של</u> N, אזי ניתן לסמן $k = m \cdot N$, ואז

$$W_N{}^{kn} = W_N{}^{mNn} = e^{j\frac{2\pi}{N}mNn} = e^{j2\pi mn} = \cos(2\pi mn) + j\sin(2\pi mn) = 1$$

אם k <u>אינה כפולה של</u> N, נסמן $q = W_N{}^k$, ואז

$$\sum_{n=0}^{N-1} 1 \cdot W_N{}^{kn} = \sum_{n=0}^{N-1} 1 \cdot q^n = \frac{q^N - 1}{q - 1} = \frac{W_N{}^{kN} - 1}{W_N{}^k - 1} = \frac{1 - 1}{W_N{}^k - 1} = \mathbf{0}$$

vectorial sum is zero



DFT: PROPERTIES EXAMPLE

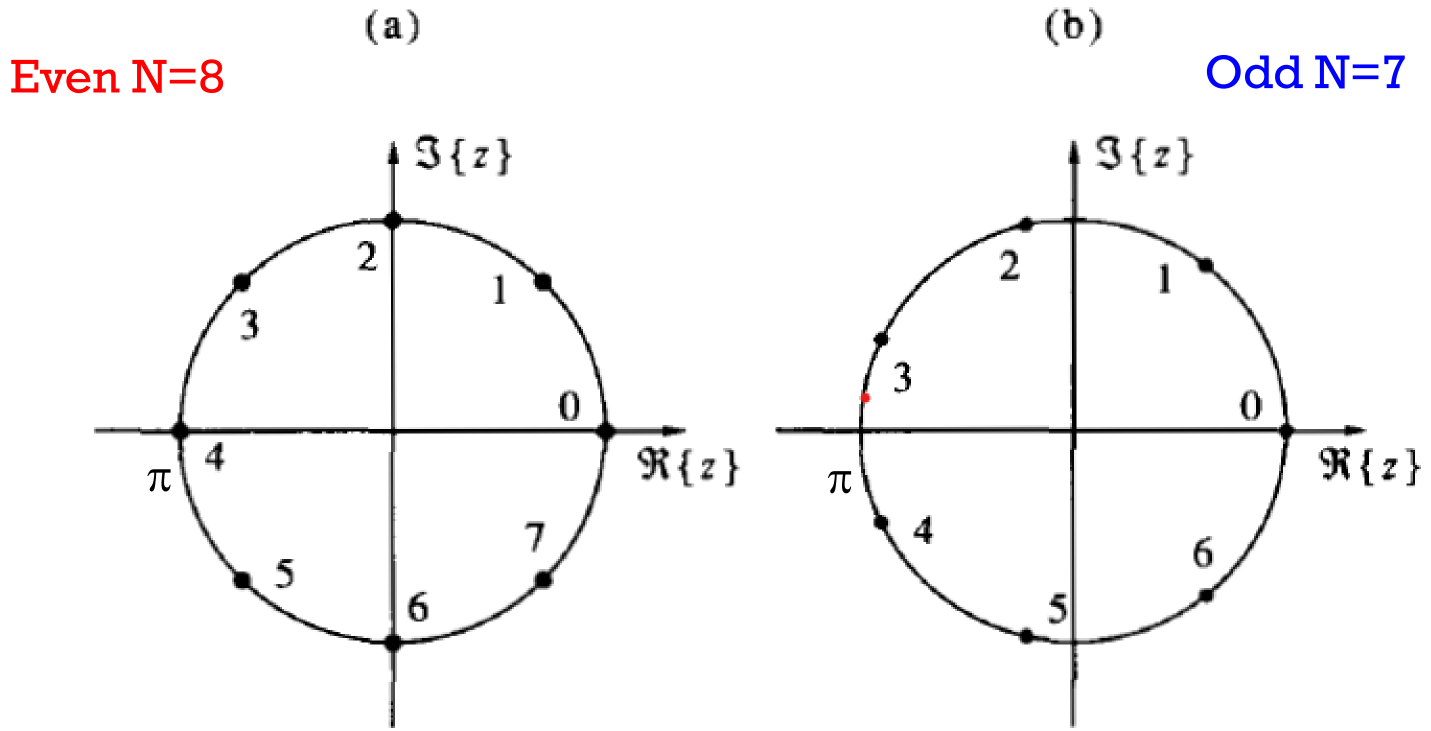


Figure 4.2 The sequence W_N^n in the complex plane: (a) even N; (b) odd N. The numbers indicate the values of n.

SUMMARY

$$\sum_{n=0}^{N-1} \left(W_N^{kn}\right) = N \cdot \delta[k \bmod N]$$

- mod = modulo $((k))_N = k - N \times \left\lfloor \frac{k}{N} \right\rfloor,\ ((k))_N$

  שארית

- In range $0 \le k \le N-1$ we get $N \cdot \delta[k]$

- Example: calculate the DFT of $x[n] = \delta[n], \quad 0 \le n \le N-1$

Solution: $$X[k] = \sum_{n=0}^{N-1} \delta(n) W_N^{-kn} = W_N^{0\mathrm{k}} = 1, \qquad 0 \le k \le N-1$$

DFT OF COS: EXAMPLE

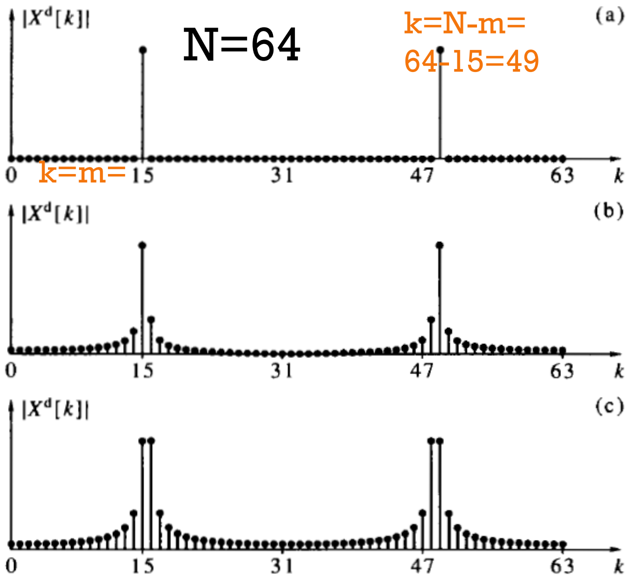


Figure 4.3 The magnitude DFT of a sinusoidal signal ($N = 64$): (a) $\theta_0 = 2\pi \cdot 15/64$; (b) $\theta_0 = 2\pi \cdot 15.25/64$; (c) $\theta_0 = 2\pi \cdot 15.5/64$.

$Assume:$ $x[n] =$ of $\cos[\theta_0 n]$, $0 \le n \le N-1$

Calculate DFT

$$X[k] = \sum_{n=0}^{N-1} \cos[\theta_0 n]\ e^{-j\frac{2\pi}{N}kn} =$$

$$(i) = \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{-jn\left(\theta_0+\frac{2\pi}{N}k\right)} + e^{jn\left(\theta_0-\frac{2\pi}{N}k\right)}\right] =$$

$\frac{q^N-1}{q-1}$ sum of geometric serries

$$(\text{ii}) = \frac{1}{2}\frac{e^{-j\theta_0 N}-1}{e^{-j\left(\theta_0+\frac{2\pi}{N}k\right)}-1} + \frac{1}{2}\frac{e^{j\theta_0 N}-1}{e^{j\left(\theta_0-\frac{2\pi}{N}k\right)}-1}$$

In case θ_0 is equal to sampling frequency of DTFT: $\theta_0 = \frac{2\pi}{N}m$ for m in range 0,..,N-1

if we insert into (i), we get $X[k] = \frac{1}{2}N \cdot \delta[\underbrace{((m+k))}_{k=N-m}{}_N] + \frac{1}{2}N \cdot \delta[\underbrace{((m-k))}_{k=m}{}_N]$ in range 0,..,N-1

DFT OF COS: EXAMPLE

Explanation:

- DTFT transform of $\cos[\theta_0 n]$, $n \in \mathbb{Z}$

$$X\left(e^{j\theta}\right) = \pi\delta(\theta - \theta_0) + \pi\delta(\theta + \theta_0)$$

- In this example the sum is of length N, therefore the sequence is not time-limited but was multiplied by a window in time. In frequency -> convolution with the Fourier transform of the window

$$\sum_{n=0}^{N-1} e^{-j\theta n} = \frac{e^{-j\theta n} - 1}{e^{-j\theta} - 1} = e^{-j\theta\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\frac{N}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

and the result is convolution between δ and "$sinc$"

When $\theta_0 \neq \frac{2\pi}{N} m$ --> X[k] are samples on "$sinc$"

When $\theta_0 = \frac{2\pi}{N} m$ --> X[k] are samples on "$sinc$“ in its zeros

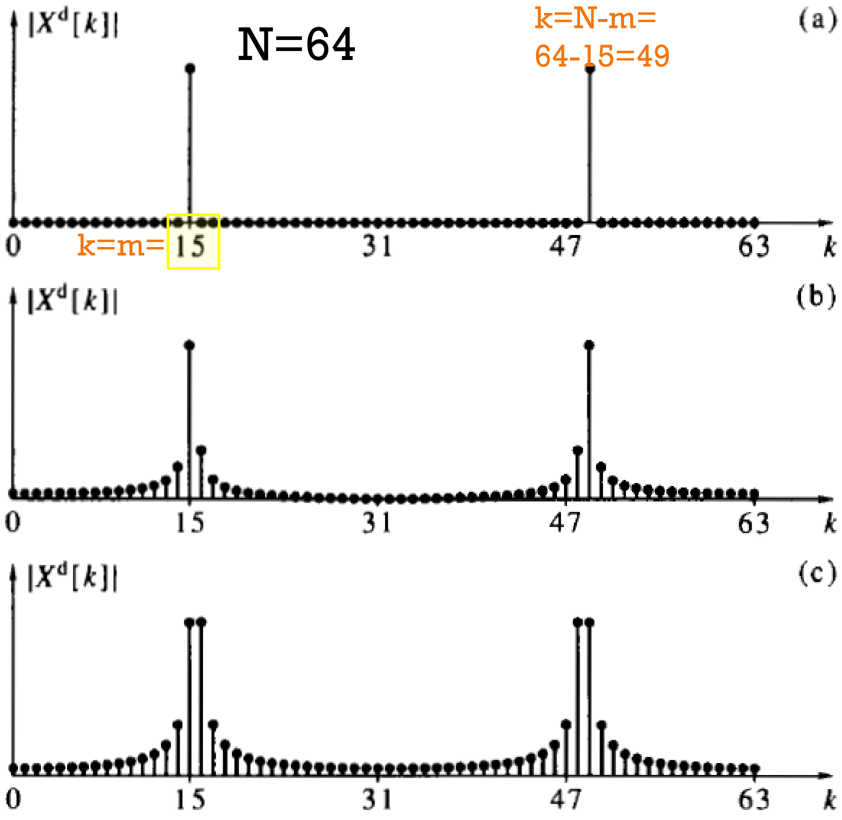
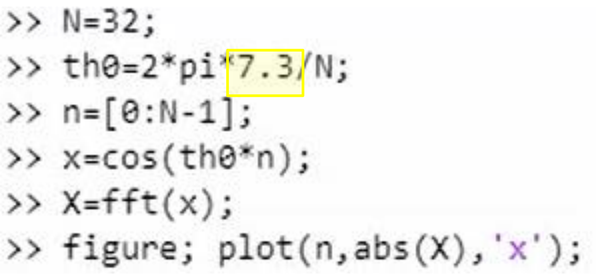
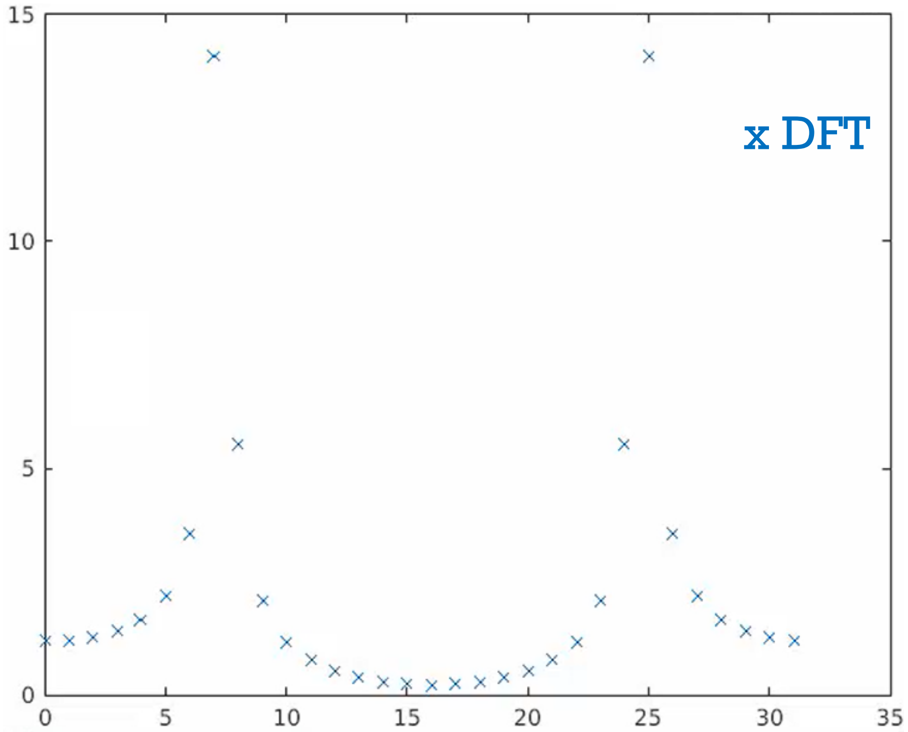


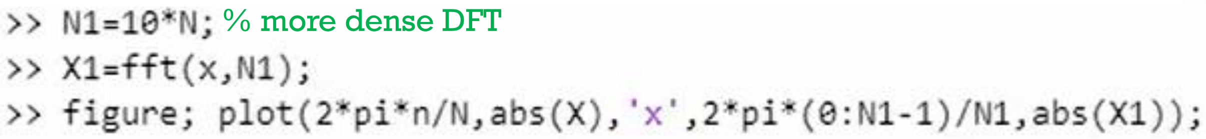
Figure 4.3 The magnitude DFT of a sinusoidal signal ($N = 64$): (a) $\theta_0 = 2\pi \cdot 15/64$; (b) $\theta_0 = 2\pi \cdot 15.25/64$; (c) $\theta_0 = 2\pi \cdot 15.5/64$.

MATLAB EXAMPLE: DTFT VS. DFT

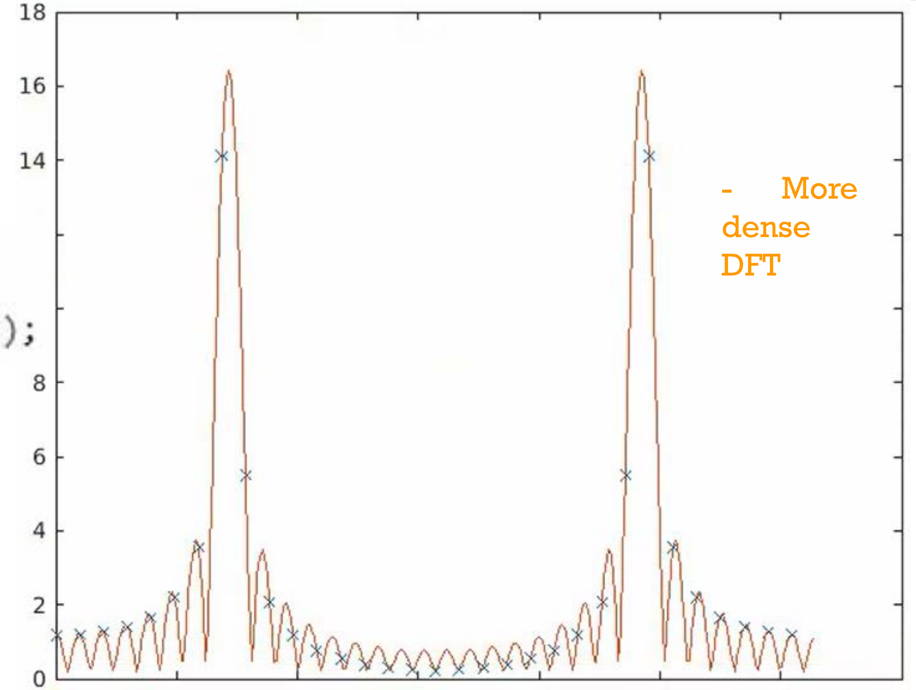


```
>> N=32;
>> th0=2*pi*7.3/N;
>> n=[0:N-1];
>> x=cos(th0*n);
>> X=fft(x);
>> figure; plot(n,abs(X),'x');
```





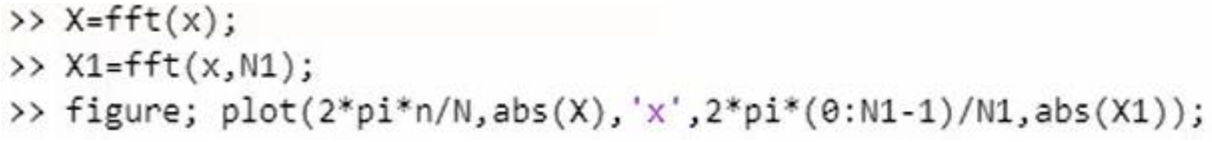
```
>> N1=10*N; % more dense DFT
>> X1=fft(x,N1);
>> figure; plot(2*pi*n/N,abs(X),'x',2*pi*(0:N1-1)/N1,abs(X1));
```



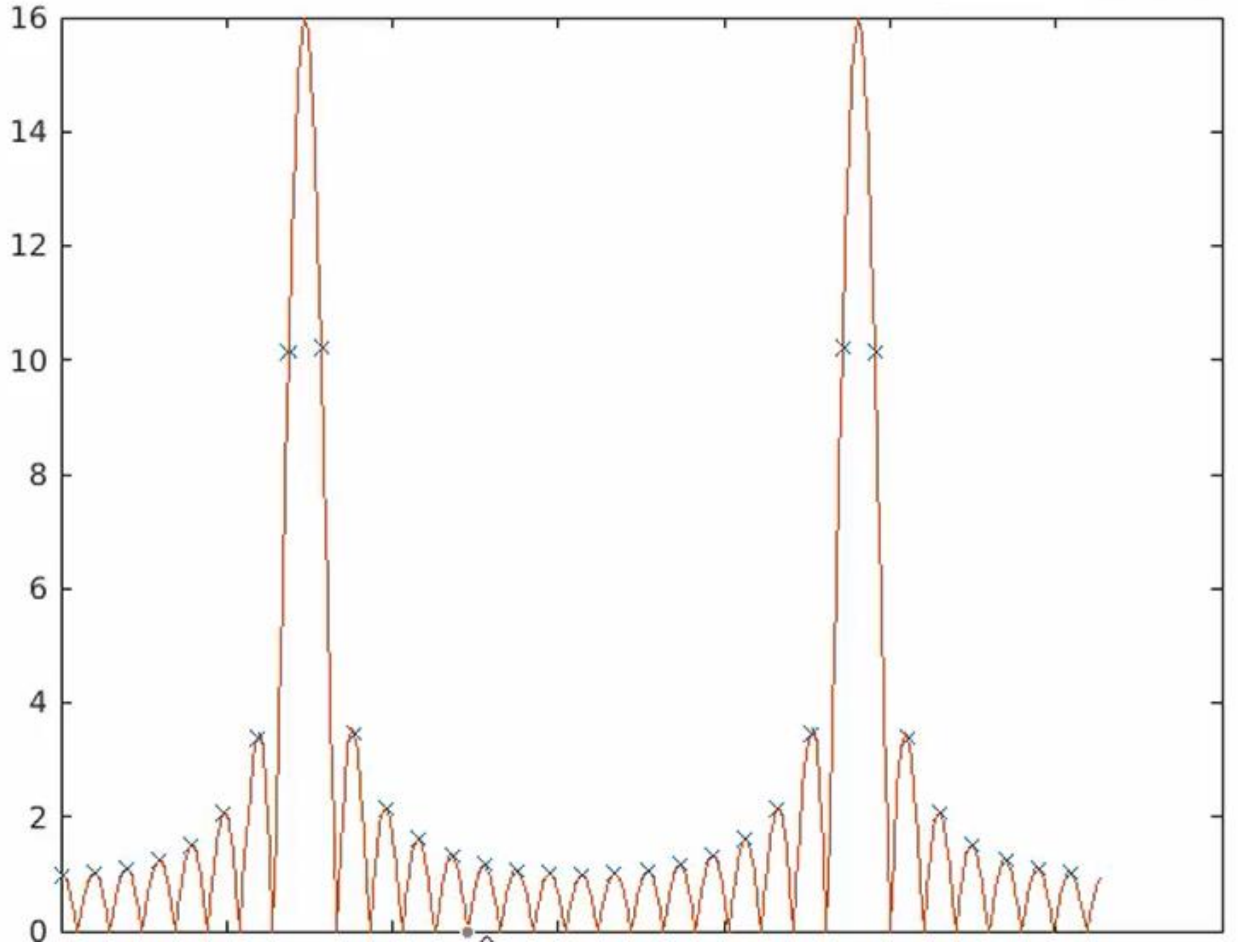
- More dense DFT

MATLAB EXAMPLE: DTFT VS. DFT

th0=2*pi*7.5/N

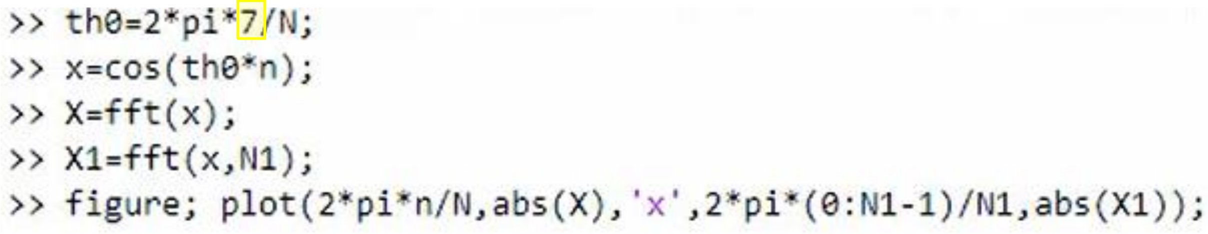


```
>> X=fft(x);
>> X1=fft(x,N1);
>> figure; plot(2*pi*n/N,abs(X),'x',2*pi*(0:N1-1)/N1,abs(X1));
```

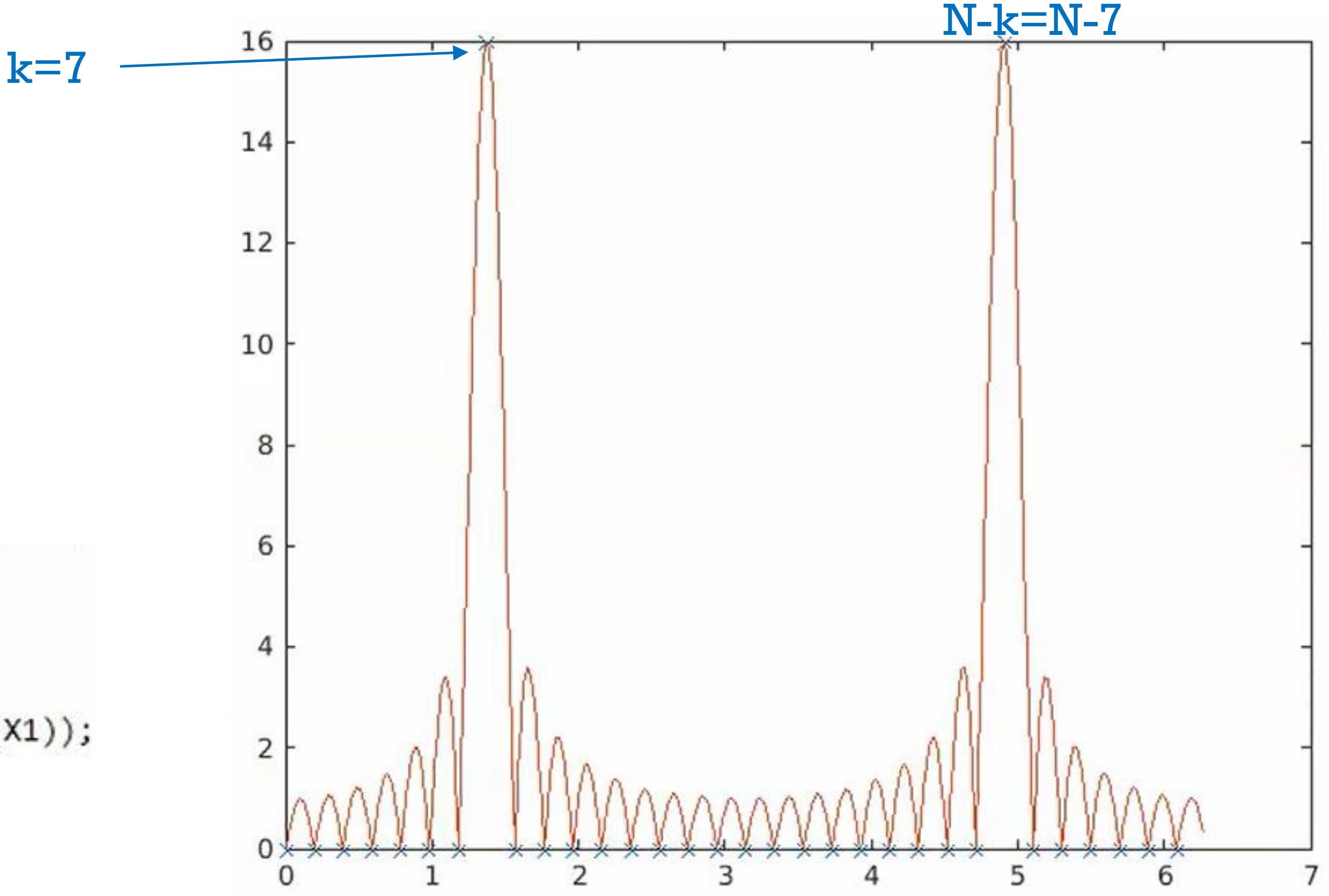


MATLAB EXAMPLE: DTFT VS. DFT

Now we choose θ_0 as a whole number of samples



```
>> th0=2*pi*7/N;
>> x=cos(th0*n);
>> X=fft(x);
>> X1=fft(x,N1);
>> figure; plot(2*pi*n/N,abs(X),'x',2*pi*(0:N1-1)/N1,abs(X1));
```



k=7

N-k=N-7

IDFT

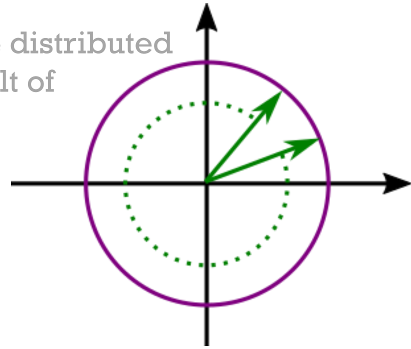
$$x[n] = IDFT_N\{X[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{kn}, \qquad 0 \le n \le N-1$$

$$W_N = e^{j\frac{2\pi}{N}} = \sqrt[\frac{1}{N}]{1}$$

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{kn} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m]W_N^{-km}\cdot W_N^{kn} =$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\underbrace{\left(\sum_{k=0}^{N-1} W_N^{k(n-m)}\right)}$$

Sum of exp=Only if points are distributed equally on unit circle the result of sum is δ



$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\overbrace{N\cdot\delta[(n-m)\bmod N]}$$

Only for $n = m \rightarrow [(n-m)\bmod N = 0, \quad 0 \le n,m \le N-1$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]N\cdot\delta[n-m] = x[n]$$

DFT - COMPLEX CONJUGATE

Let $y[n] = x[n]^*$

Therefore DFT{ $y[n]$} $= Y[k] = X^*[N-k]$

Proof:

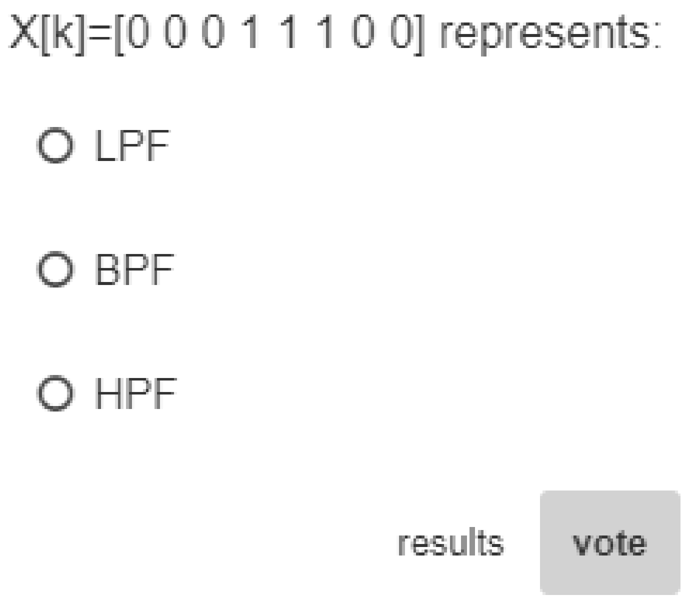
$$\underline{Y[k]} = \sum_{n=0}^{N-1} x^*[n]W_N^{-kn} = \left(\sum_{n=0}^{N-1} x[n]W_N^{kn}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} x[n]W_N^{-(-k)n}\right)^* = \left(\sum_{n=0}^{N-1} x[n]W_N^{-(N-k)n}\right)^*$$

$$= \underline{X^*[N-k]}$$

SURVEY: DFT TRANSFORM

- EasyPolls:



X[k]=[0 0 0 1 1 1 0 0] represents:

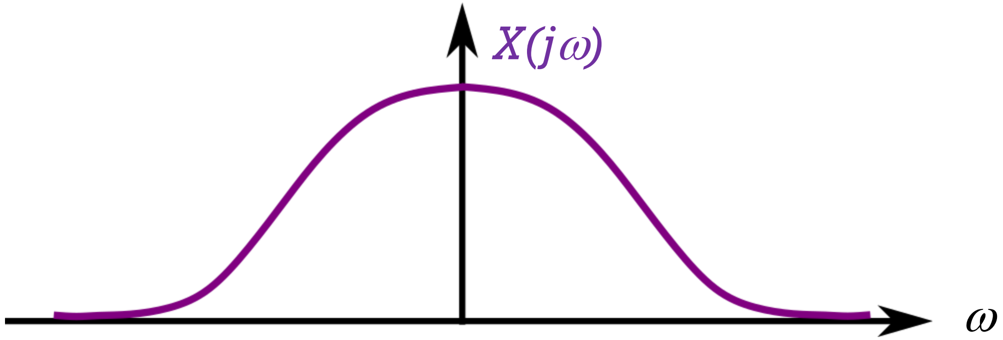
○ LPF

○ BPF

○ HPF

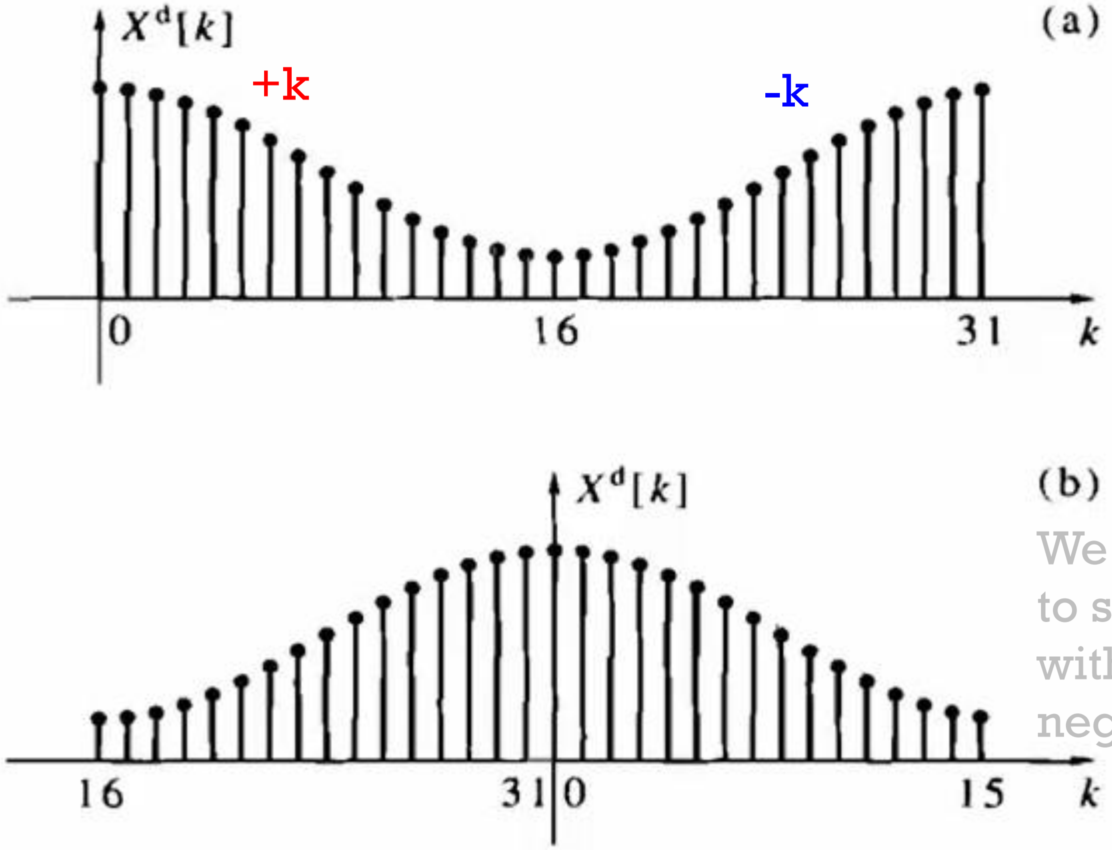
DFT: FREQUENCY REPRESENTATION

- With Fourier Transform (FT) we were representing ***X(jω)*** via positive and negative frequencies.



- In DTFT we showed for the period $\theta \in [-\pi, \pi]$ while $\pi \;\; equivalent \;\; \rightarrow \frac{\omega_s}{2}$ is the half frequency
- In DFT, k=0,….,N-1 $\theta_k = \frac{2\pi}{N} k \in [0{,}2\pi]$ and therefore negative frequencies appear to the right while positive frequencies fulfill $0 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor$

DFT: FREQUENCY REPRESENTATION

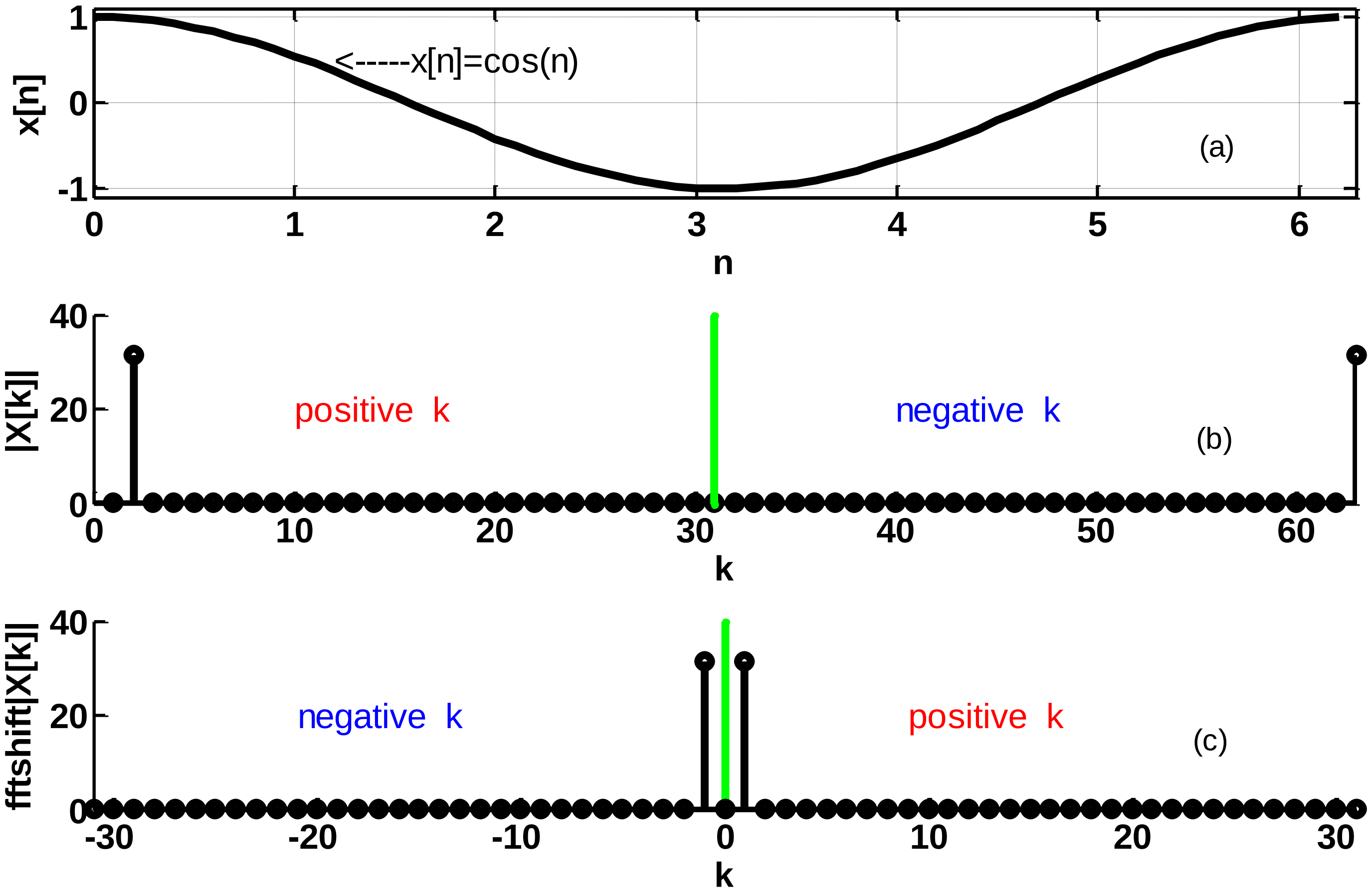


We used
to see something like this
with positive to the right and
negative to the left frequencies

Figure 4.4 Rearrangement of the DFT: (a) index k in original order; (b) index k in a shifted order (shown for $N = 32$).

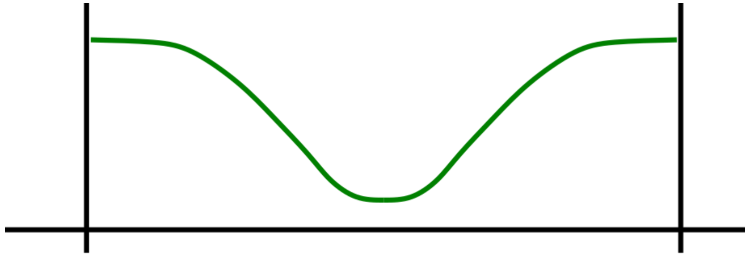
DFT OF COS: MATLAB EXAMPLE

DFT for
N=64

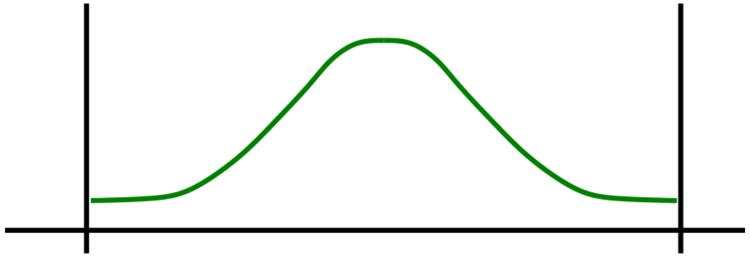


DFT: FREQUENCY REPRESENTATION

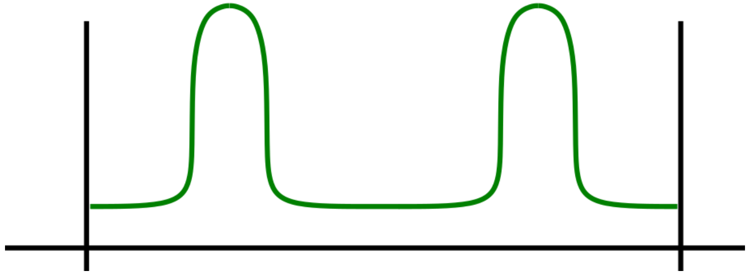
LPF

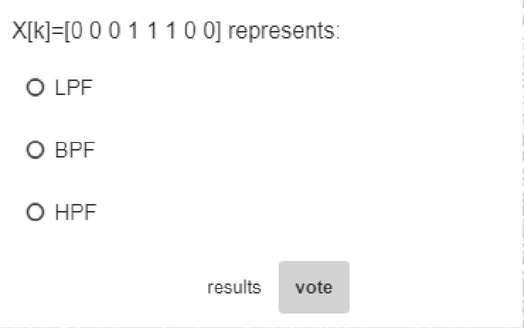


HPF



BPF





X[k]=[0 0 0 1 1 1 0 0] represents:

○ LPF

○ BPF

○ HPF

DFT PROPERTY: SYMMETRY

- Let assume signal $x[n]$ as real

Proof that $X[k] = X^*[N-k]$

<u>Proof</u>:

$$X^*[N-k] = \sum_{n=0}^{N-1} \left[x[n]e^{-j(N-k)\frac{2\pi}{N}n}\right]^*$$

$$= \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}e^{jN\frac{2\pi}{N}n} = 1$$

$$= X[k]$$

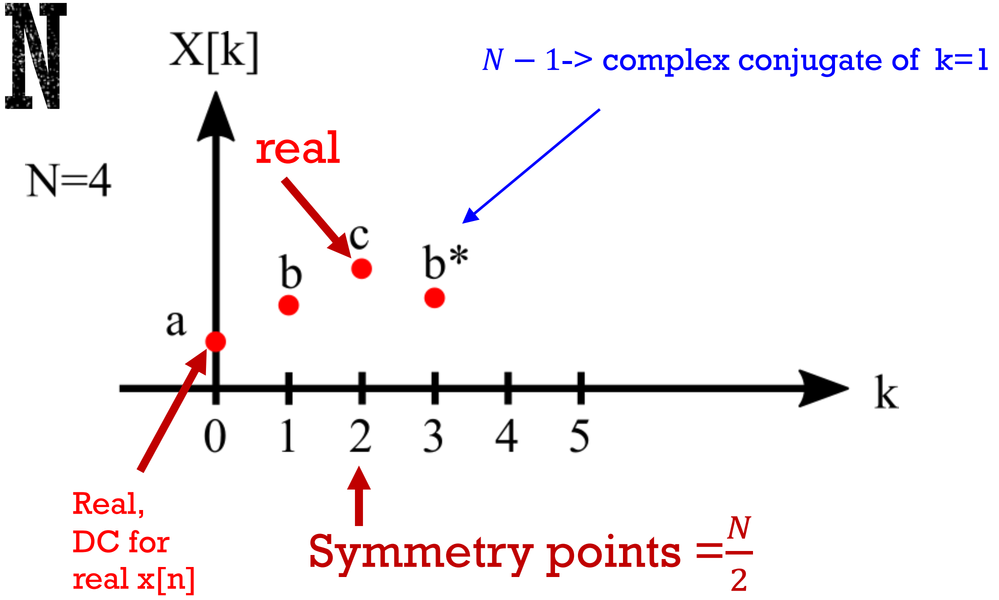
Meaning:

1) One can calculate only samples $k = 0 \ldots N/2$ (N is even) or $k = 0 \ldots (N-1)/2$ (N is odd)
2) If N is even then X[N/2] is real.

EXAMPLE: DFT INTUITION

Let assume that we can illustrate
on the same plot real and imaginary values

Illustrate a, b, c, points for different length N of DFT

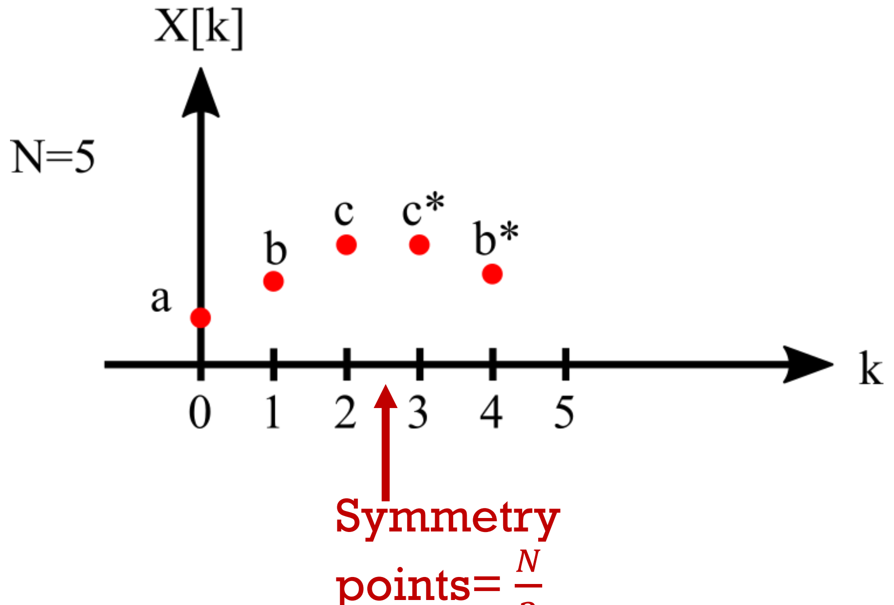


$N-1$-> complex conjugate of k=1

real

Real,
DC for
real x[n]

Symmetry points $=\frac{N}{2}$



Symmetry
points= $\frac{N}{2}$