FIR LINEAR PHASE RESPONSE, SUMMARY

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Reading: Chapters 9, 10 by Proakis and Monolakis; Chapter 7.7 by Opp& Shafer

FIR – LINEAR PHASE

- **Linear Phase** or **Delay** (השהיה) is wanted since it is not distorting the signal.

- For filters symmetry around the center of the filter

$$h[n] = \pm h[M-1-n]\,, \qquad n = 0\,,\,\dots\,,\,M-1$$

the phase is linear.

$$h(n) = \pm h(M-1-n) \qquad (9.2.5)$$

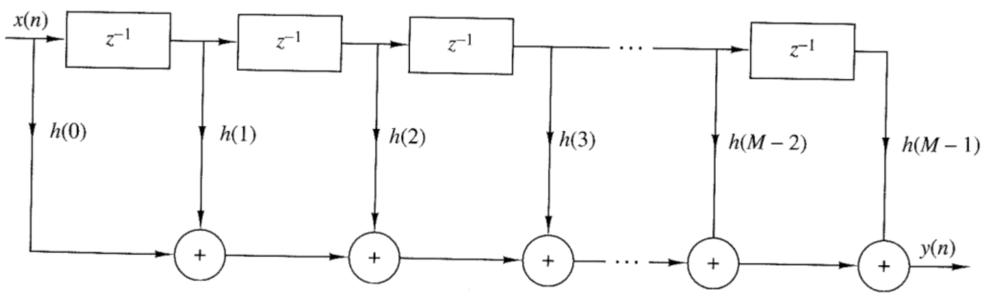


Figure 9.2.1 Direct-form realization of FIR system.

FIR – LINEAR PHASE

- Why is the linear phase needed?
- **Linear phase – delay**. The signal is only delayed and there is no additional distortion or another change in a signal.
- We will show that for FIR that fulfill the following equality, there is a linear phase:

$$h[n] = \pm h[M-1-n]\,, \qquad n = 0\,,\ \dots\,,\ M-1$$

Therefore, there is a symmetry around the center of the filter. Let prove that:

Let M-even, we will calculate $H(z)$:

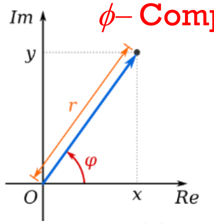
$$H(z) = \sum_{n=0}^{M-1} h[n]z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]z^{-n} + \sum_{n=\frac{M}{2}}^{M-1} h[n]z^{-n}$$

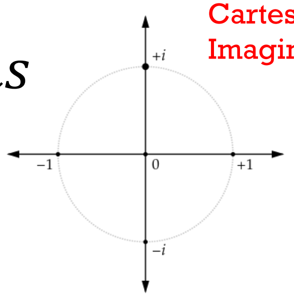
separate to two summations

$$z = Ae^{j\phi} = A \cdot (\cos\phi + j\sin\phi)$$

ϕ– Complex argument



j in complex or Cartesian plane= Imaginary unit



FIR – LINEAR PHASE

- Substitute $n = M - 1 - n'$ in the second summation:

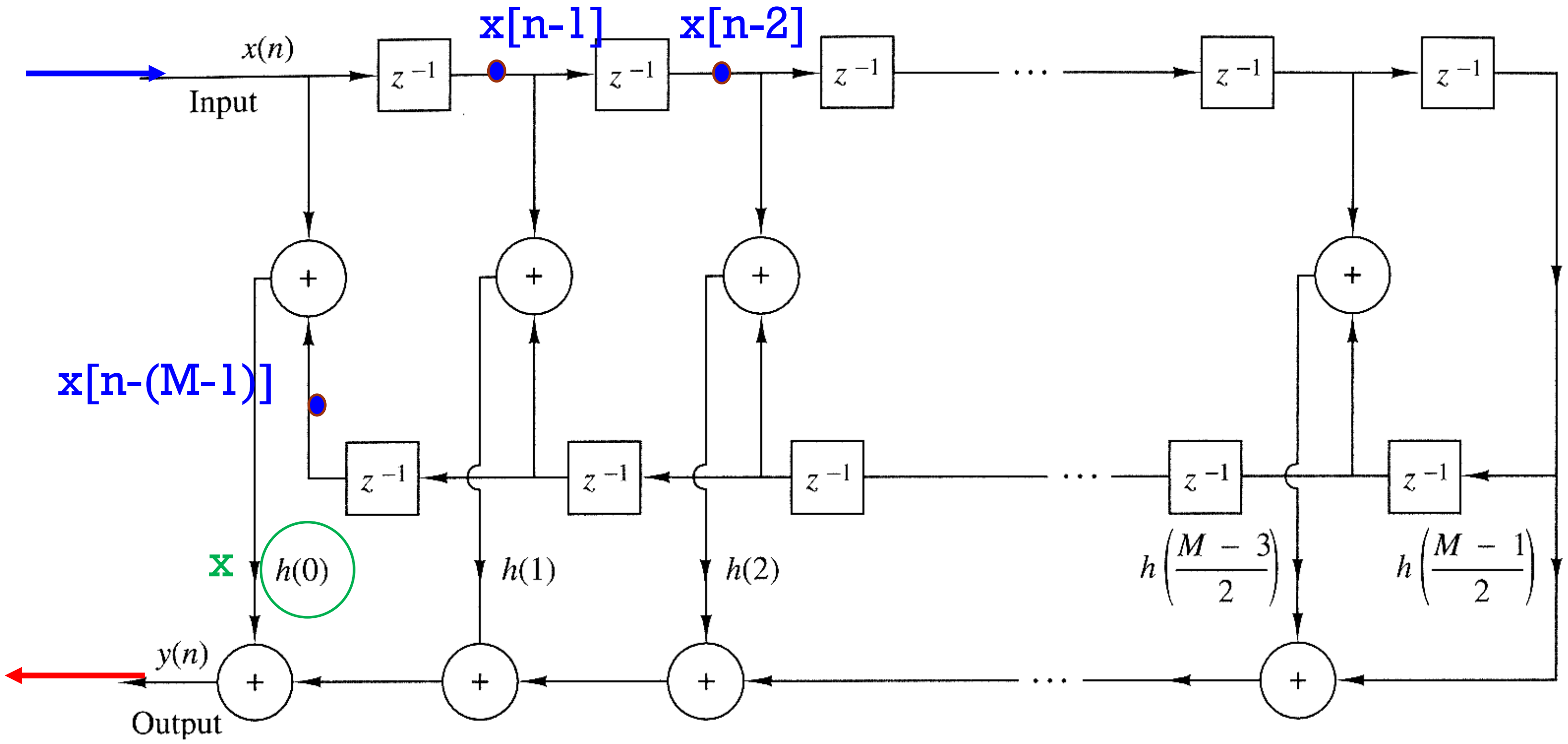
$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]Z^{-n} + \sum_{n=0}^{\frac{M}{2}-1} h[M-1-n']Z^{-(M-1-n')}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[\mathrm{n}]\left[Z^{-n} \pm Z^{-(M-1-n)}\right]$$

- We will merge to sums, substitute the symmetry. From this result we see that we can anticipate multiplications in calculations see Figure in the next slide

FIR – LINEAR PHASE

For such a system the number of multiplications is reduced from M to M/2 for M even and to (M-1)/2 for M odd, for instance as here:



Schematic representation
Of linear phase FIR
filters

Figure 9.2.2 Direct-form realization of linear-phase FIR system (M odd).

FIR – LINEAR PHASE

Now we will analyze the phase $<H\left(e^{j\theta}\right)$ and therefore we will work with **DTFT** via substitution of z=$H\left(e^{j\theta}\right)$:

$$H\left(e^{j\theta}\right)=\sum_{n=0}^{\frac{M}{2}-1} h(n)\left[e^{-j\theta n}\pm e^{-j\theta(M-1)}e^{j\theta n}\right]$$

We take out the half angle as a joint component:

$$=\sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-j\theta\frac{M-1}{2}}\left[e^{-j\theta\left(n-\frac{M-1}{2}\right)}\pm e^{j\theta\left(n\frac{M-1}{2}\right)}\right]$$

for + we will receive:

$$=2e^{-j\theta\frac{M-1}{2}}\underbrace{\sum_{n=0}^{\frac{M}{2}-1} h[n]\cos\left(\theta\left[n-\frac{M-1}{2}\right]\right)}_{A(\theta)}$$

$A(\theta)$ linear phase, delay in (M-1)/2 samples – not an integer

$A(\theta)$ is the real function, Amplification
Is it equal to |H|?
No, because it can be negative, phase shift in π

FIR – LINEAR PHASE

When $A(\theta)$ is positive we will obtain the linear phase.

When $A(\theta)$ is negative we will obtain the phase inversion or shift in π at the points the sighn experiences the change from – to +

(piecewise phase) פאזה ליניארית למקוטעין
segmented phase

SURVEY: A FILTER PROPERTIES

- EasyPolls:

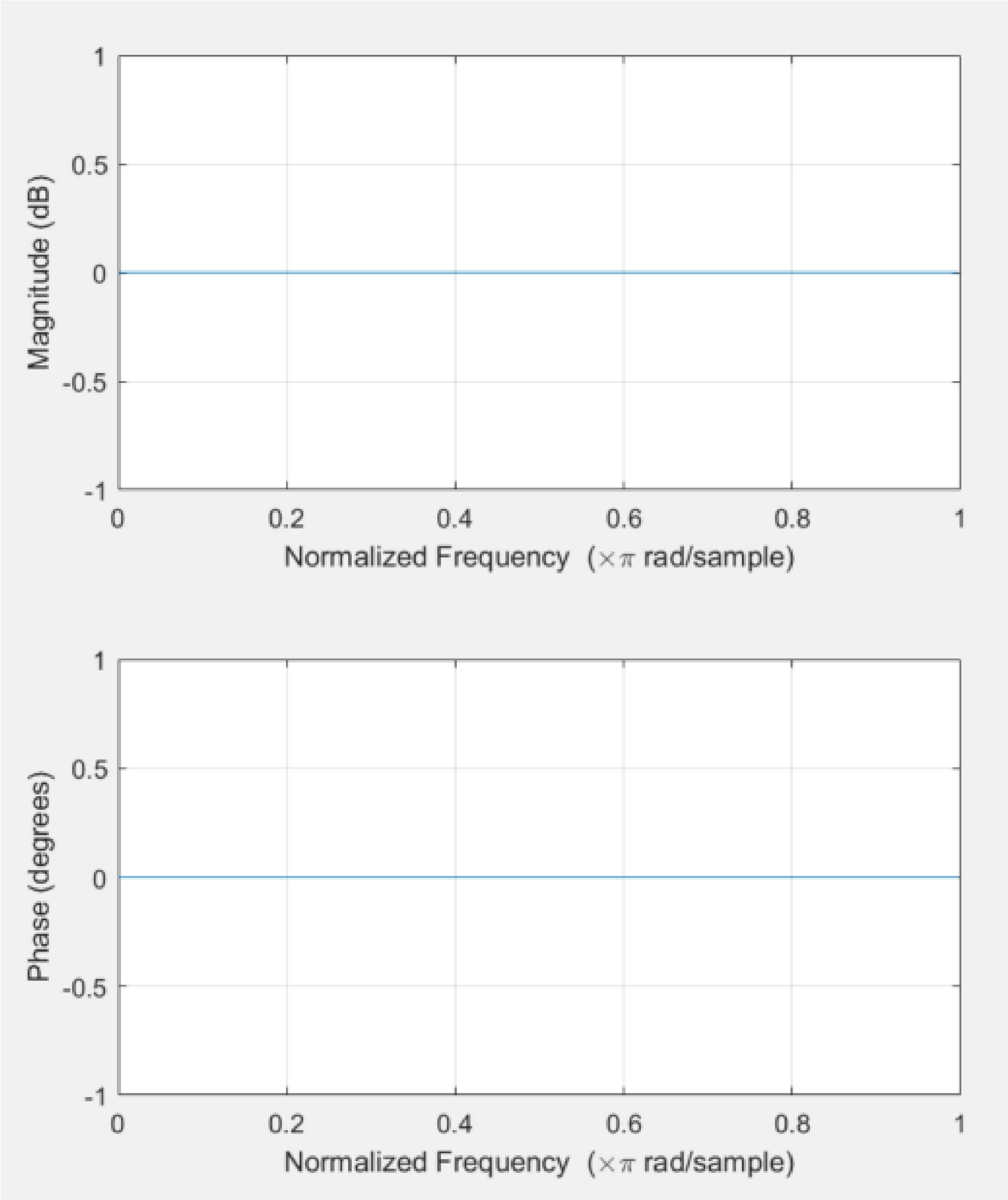
h[n] is an FIR filter of length M. It is known that the symmetry causes linear phase. Does the vise versa valid: does the linear phase mean symmetry of the filter? Note: symmetry means h[n]=+-h[M-1-n].

- ○ Yes
- ○ No

results vote

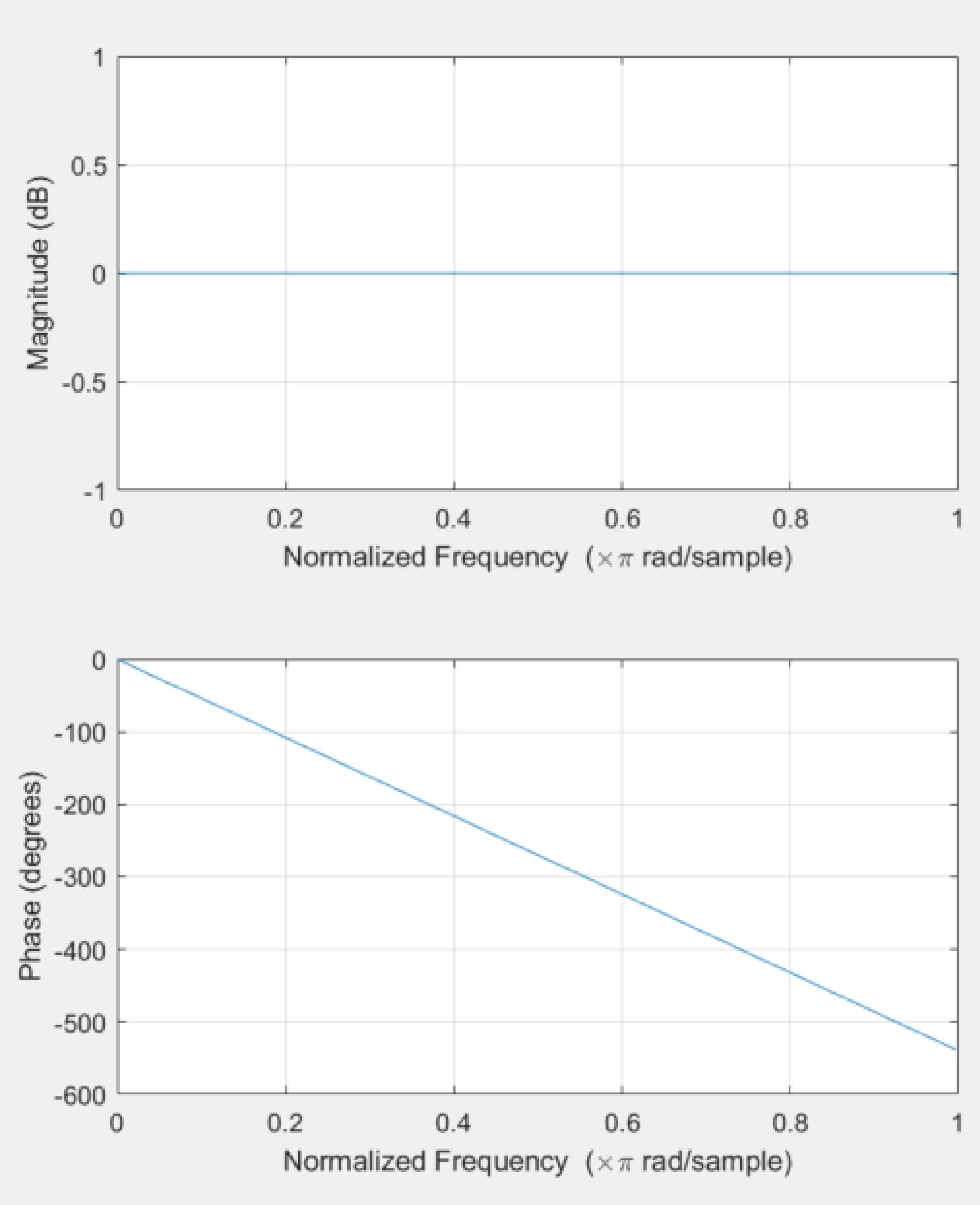
MATLAB FIR

```
>> h=[1];
>> freqz(h,1,300)
```



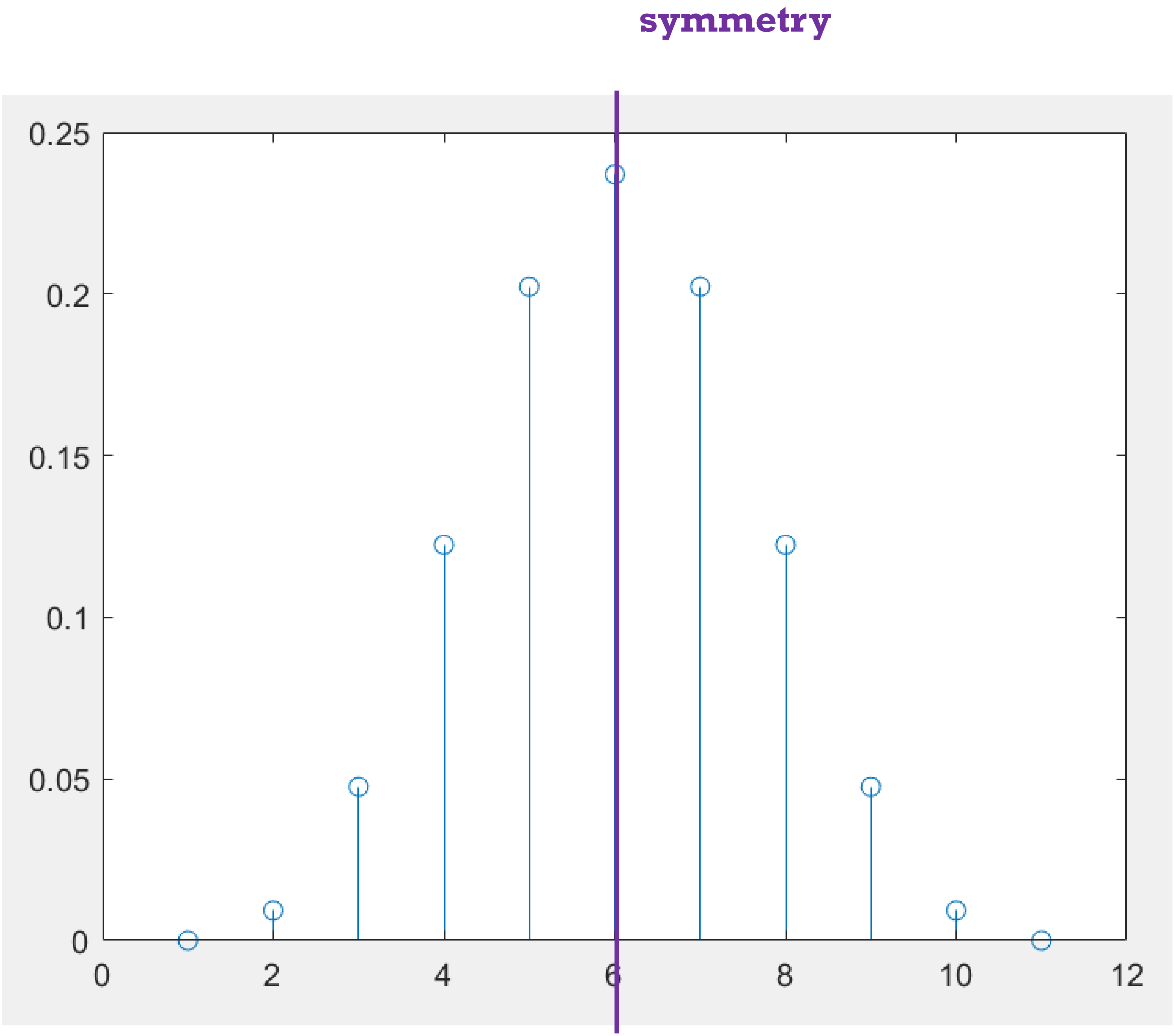
MATLAB FIR

```
>> h=[0 0 0 1];
>> freqz(h,1,300)
```



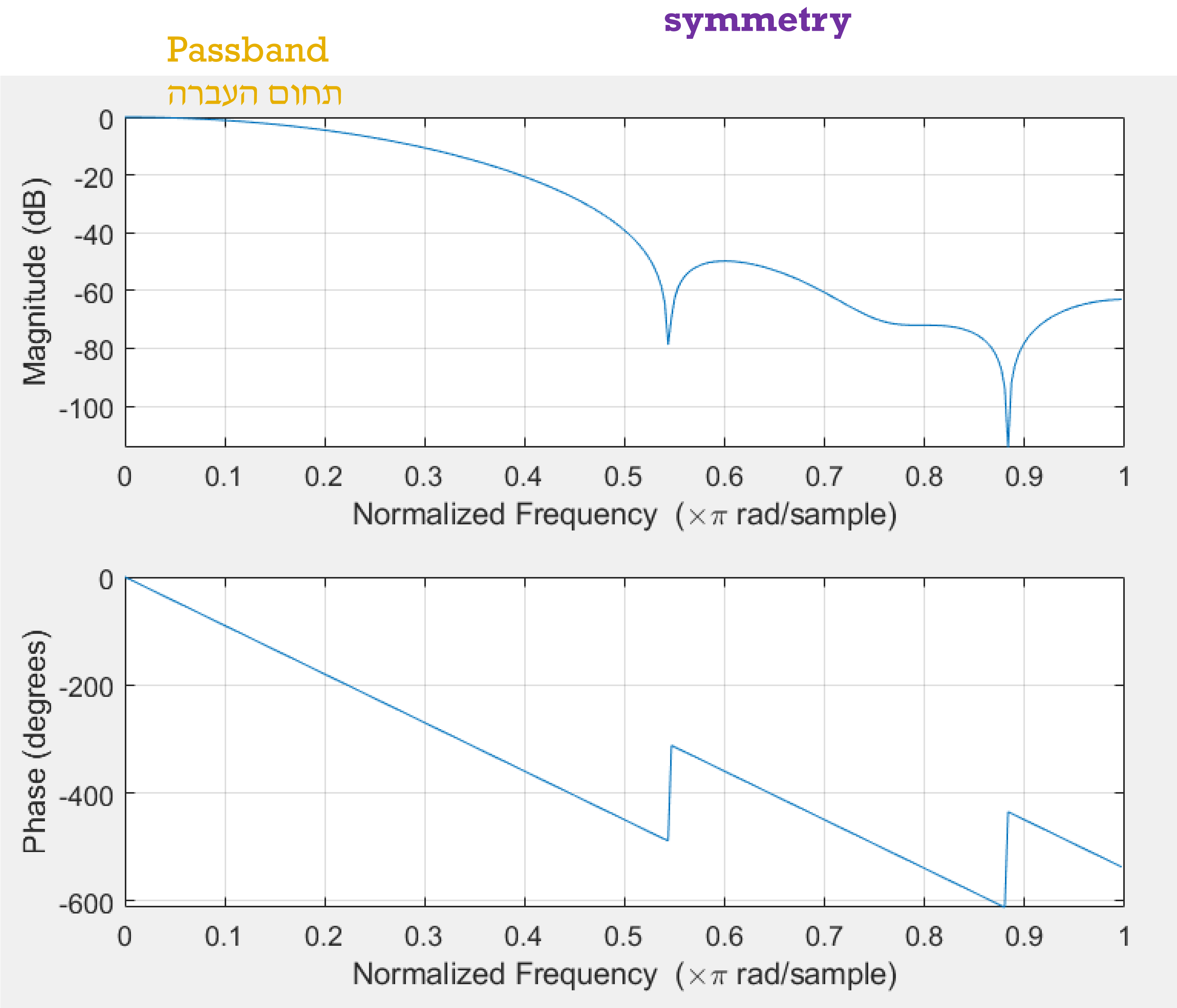
MATLAB FIR

```
>> h=fir1(10,0.2);
>> stem(h)
```



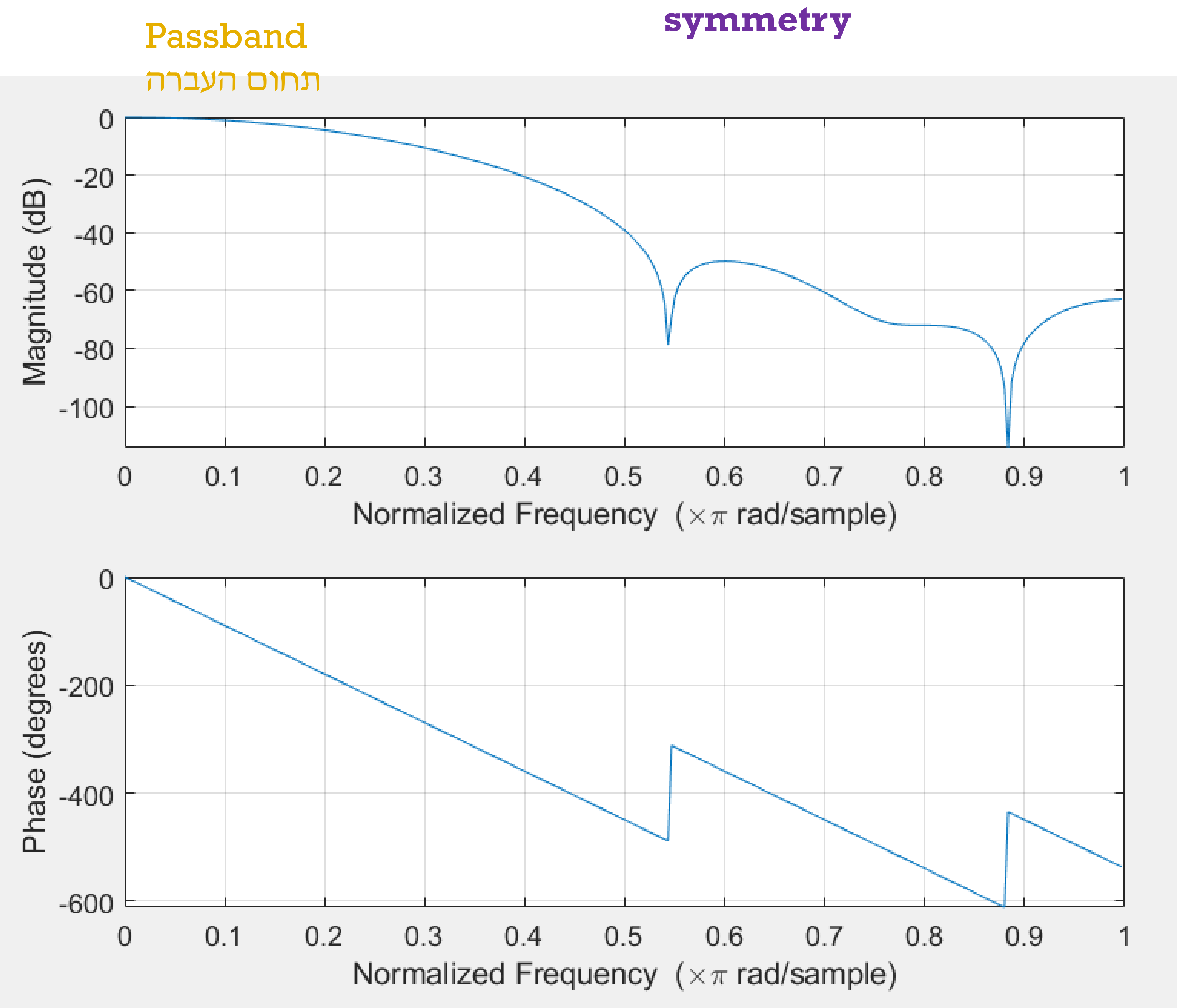
MATLAB FIR

```
>> h=fir1(10,0.2);
>> freqz(h,1,300)
```



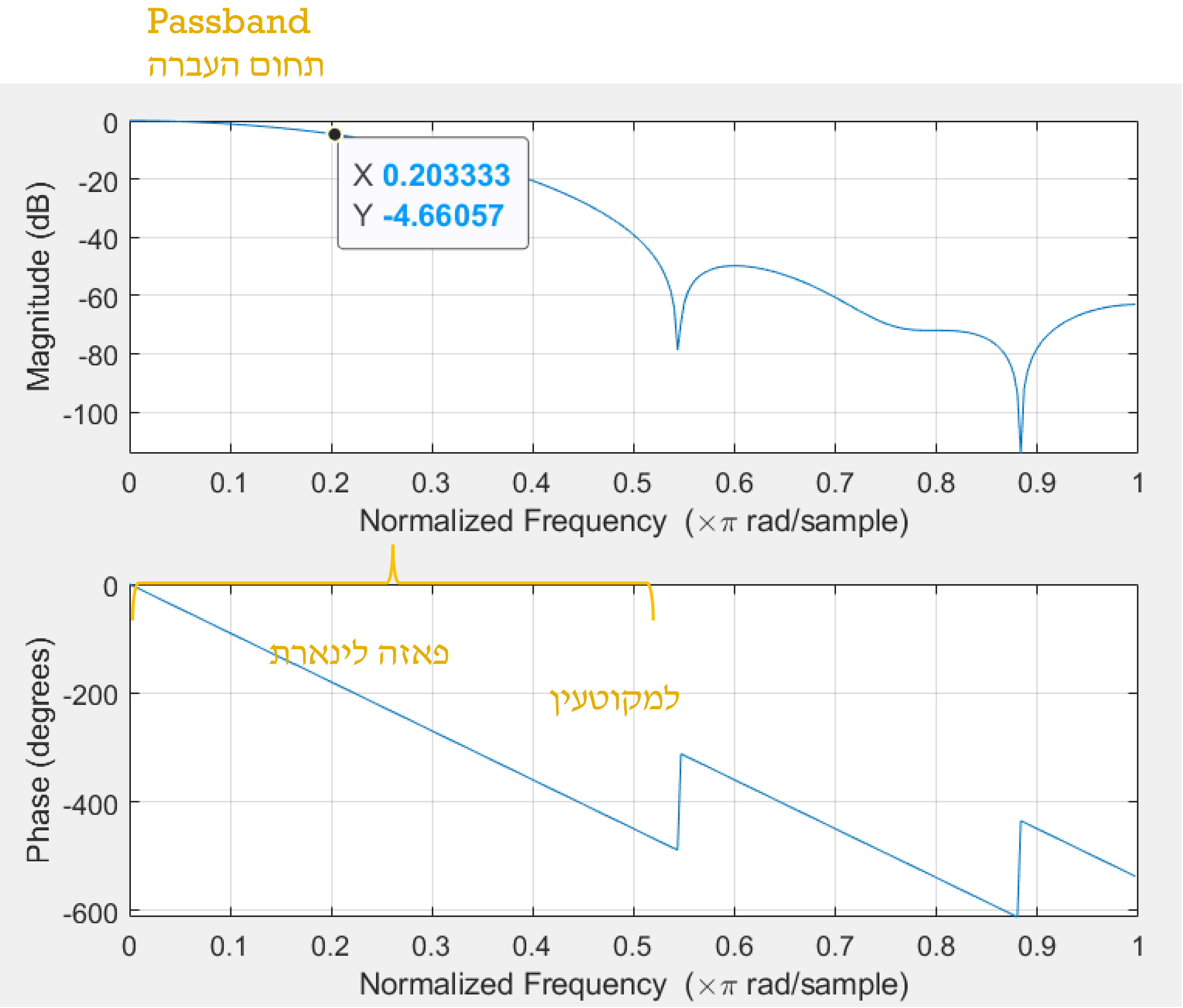
MATLAB FIR

```
>> h=fir1(10,0.2);
>> freqz(h,1,300)
```



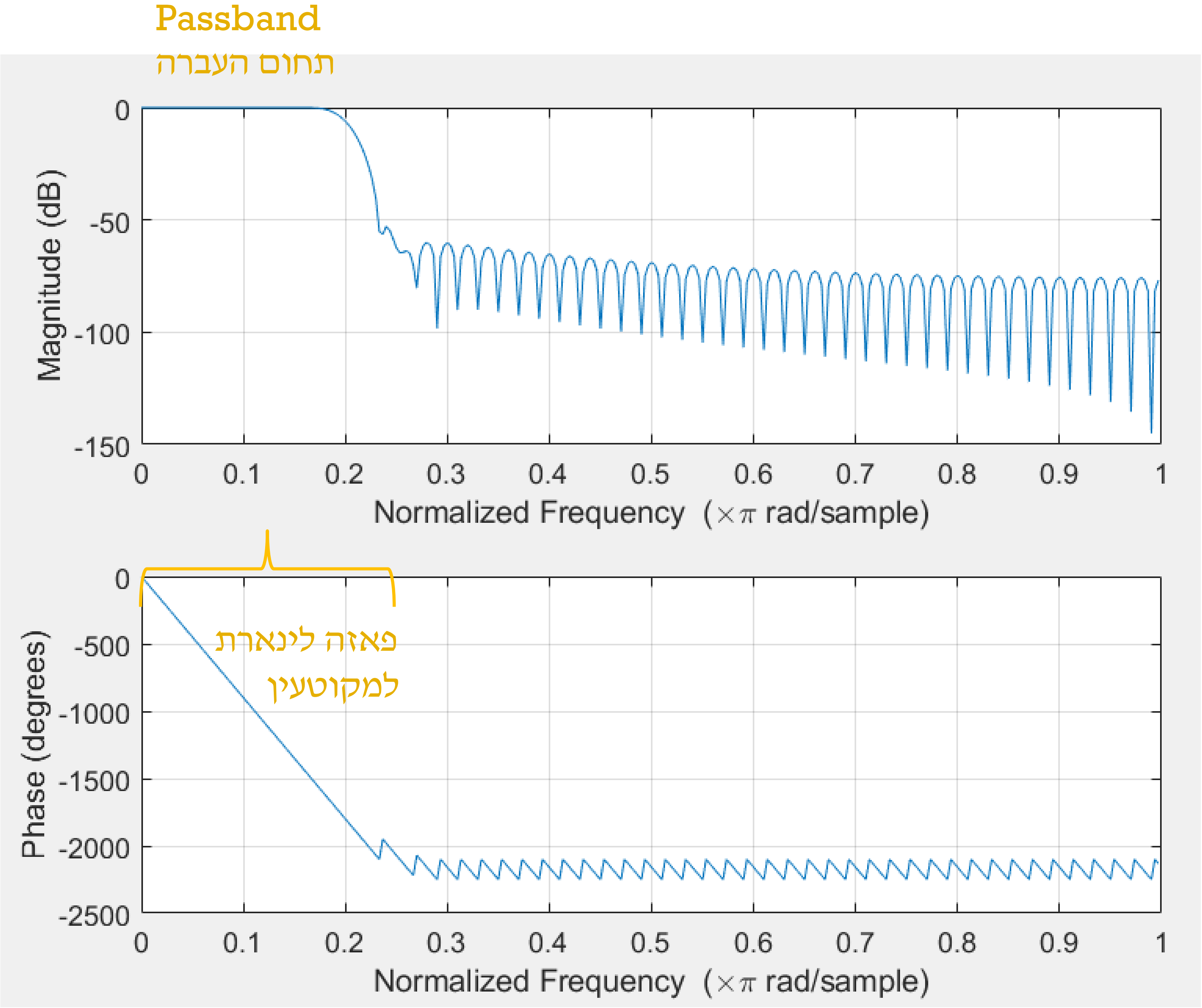
MATLAB FIR

```
>> h=fir1(100,0.2);
>> freqz(h,1,300)
```



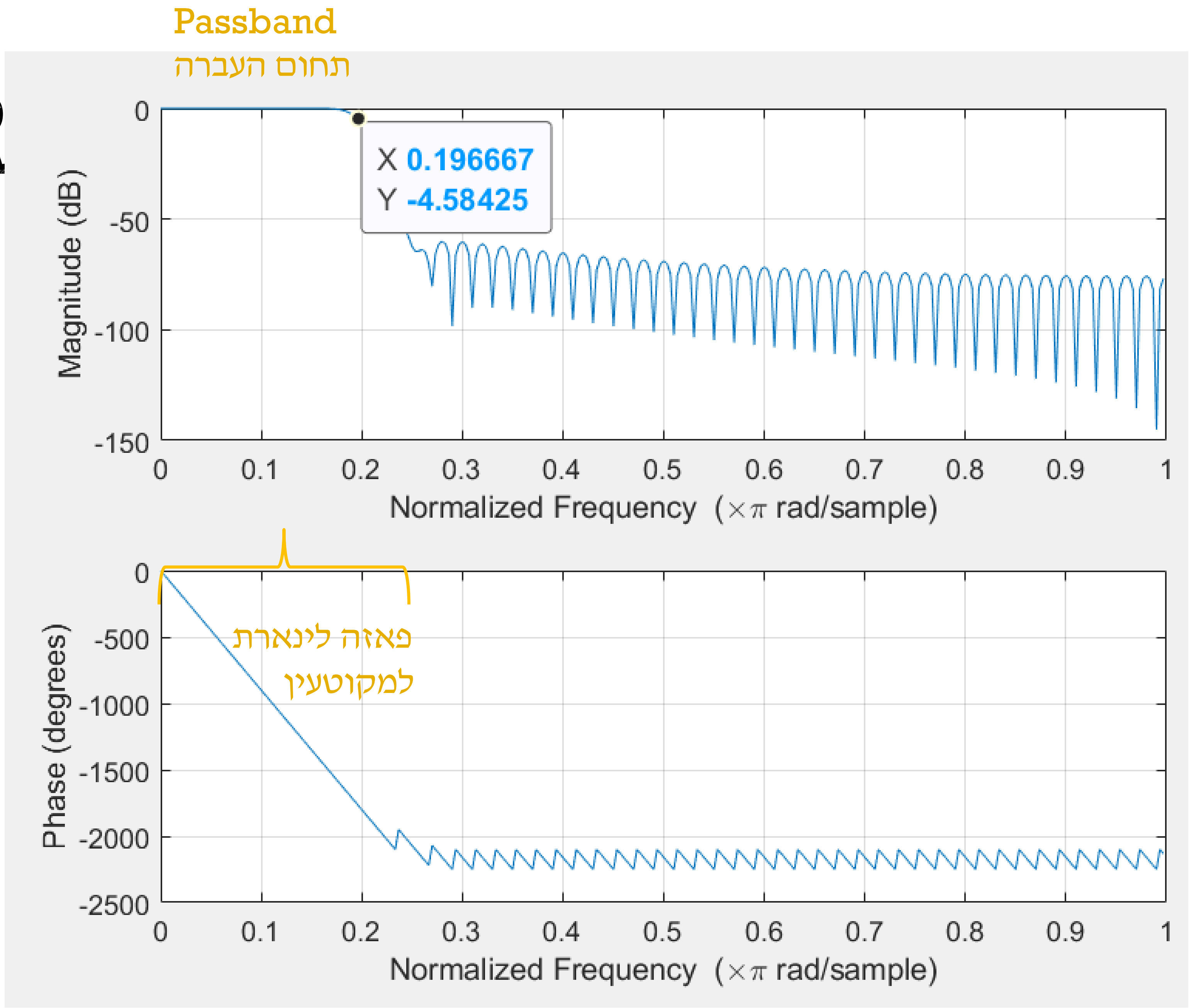
MATLAB FIR

```
>> h=fir1(100,0.2);
>> freqz(h,1,300)
```



MATLAB FIR

```
>> h=fir1(100,0.2);
>> freqz(h,1,300)
```

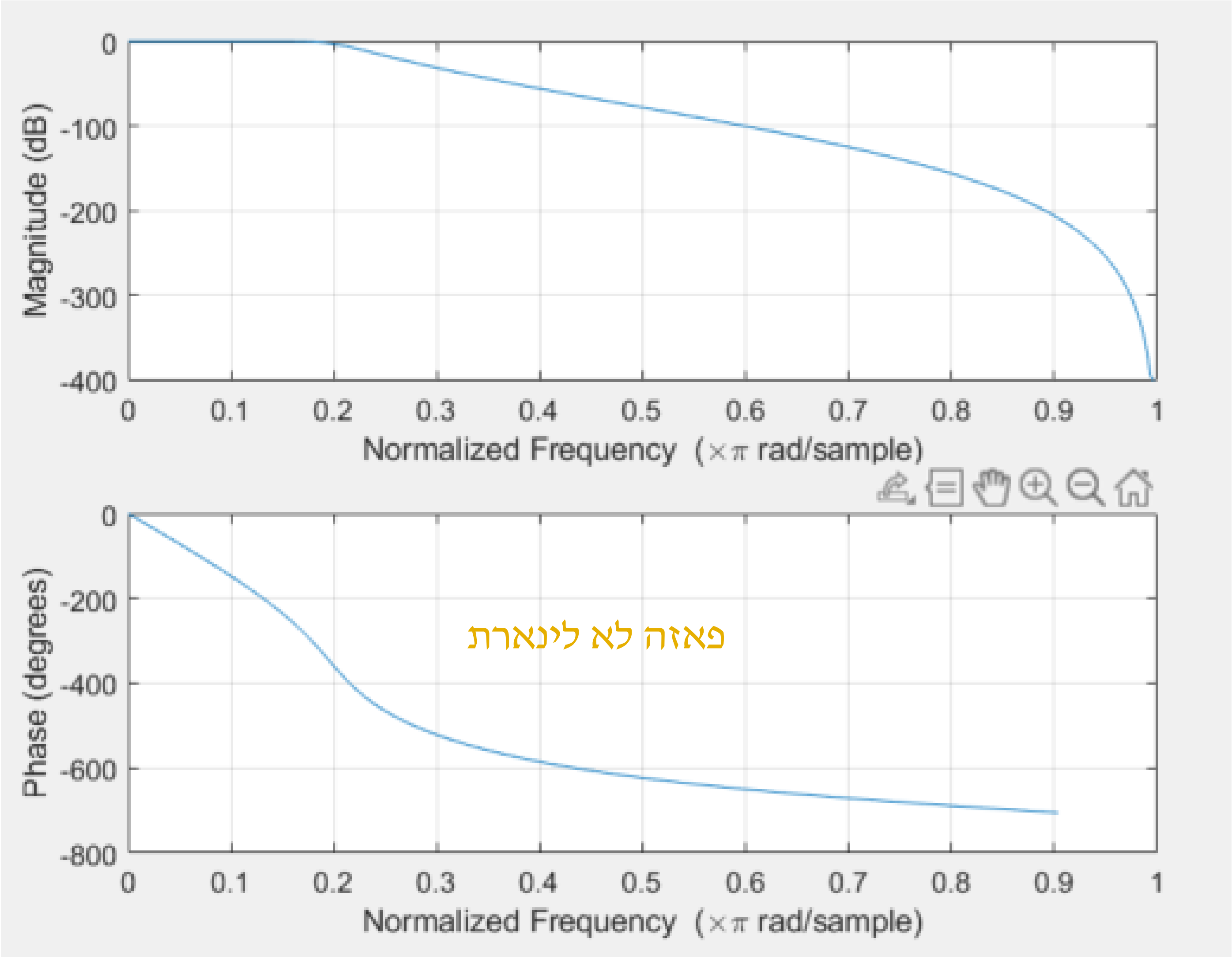


MATLAB IIR

```
>> [b,a]=butter(8,0.2);
>> freqz(b,a,300)
```

Passband

תחום העברה

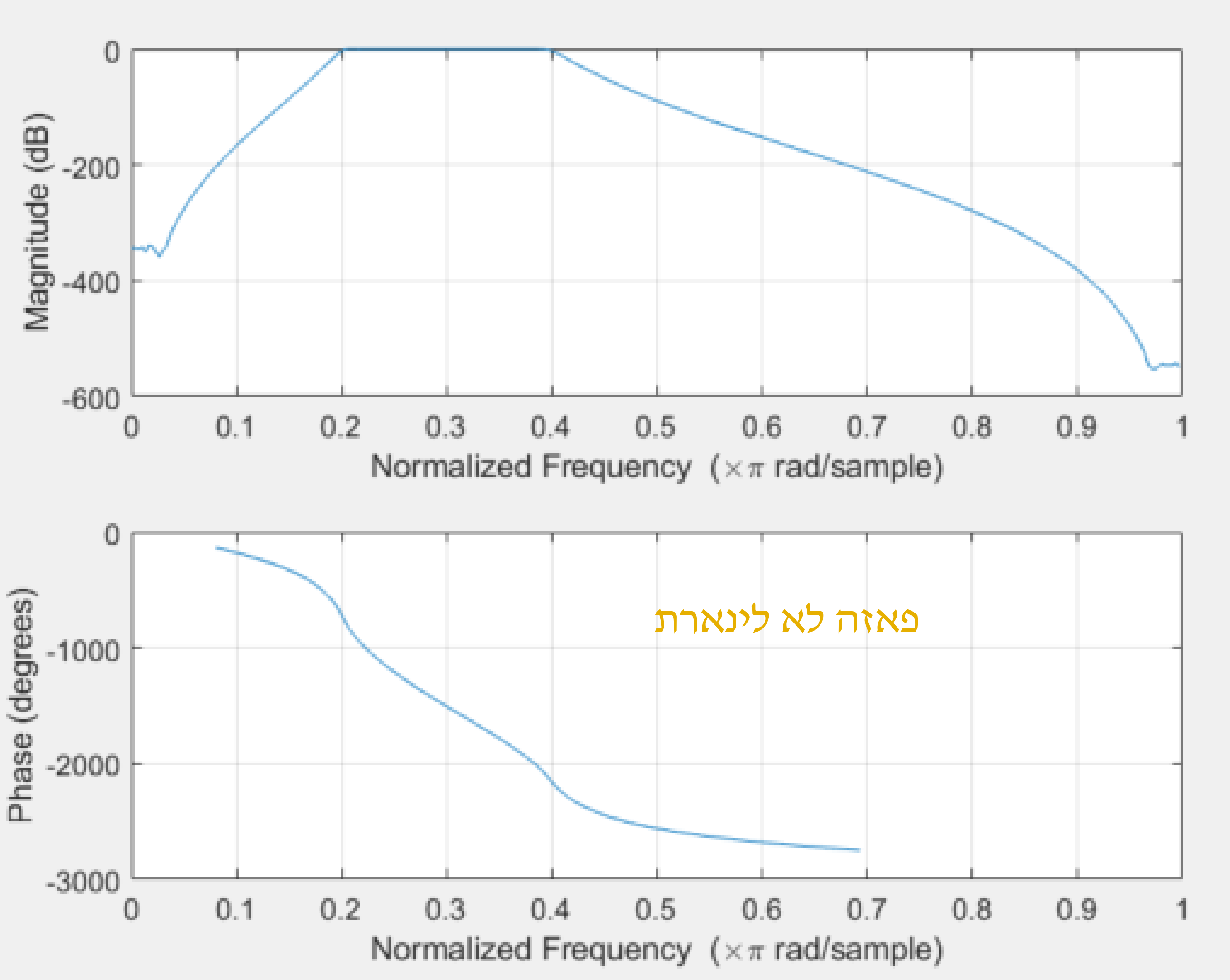


MATLAB IIR

```
>> [b,a]=butter(16,[0.2,0.4]);
>> freqz(b,a,300)
```

Passband

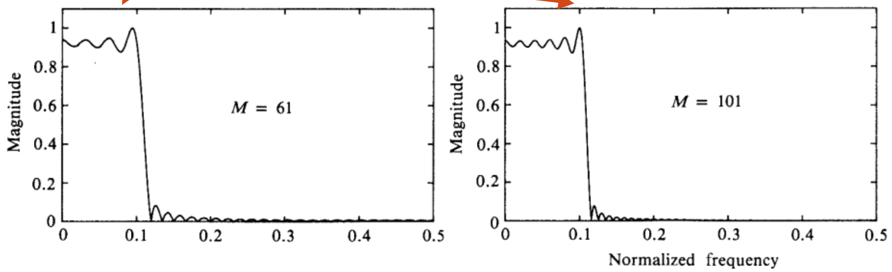
תחום העברה



פאזה לא לינארית

FILTERS: DEFINITION

Gibbs phenomenon



Ripple= גליות/אדווה

Occurs due to the finite length sequence. The longer the sequence less ripples

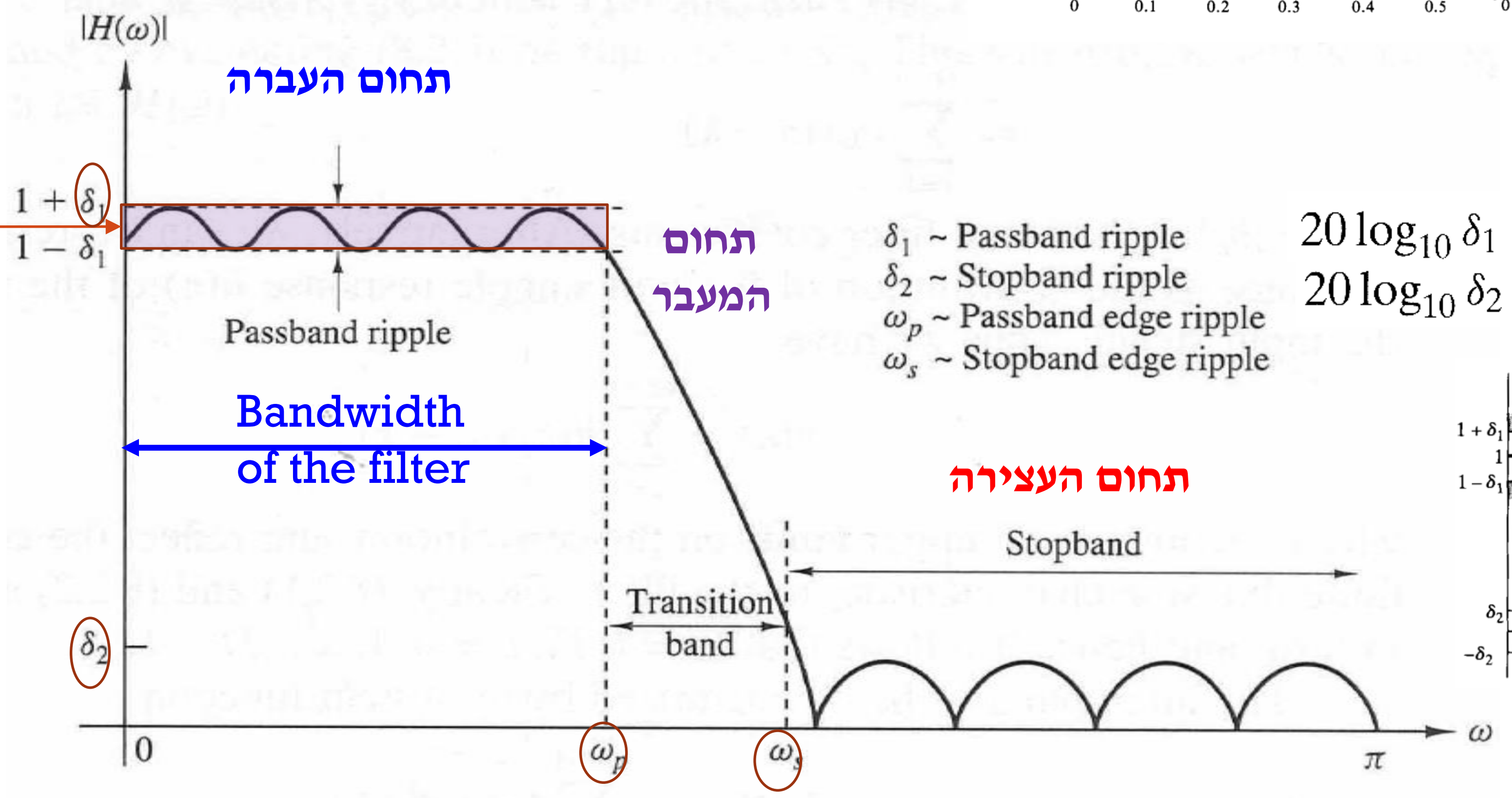


Figure 8.2 Magnitude characteristics of physically realizable filters.

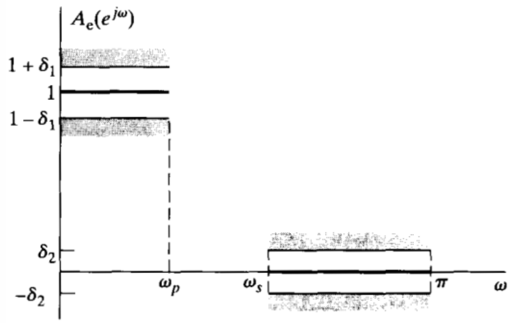


Figure 7.31 Tolerance scheme and ideal response for lowpass filter.

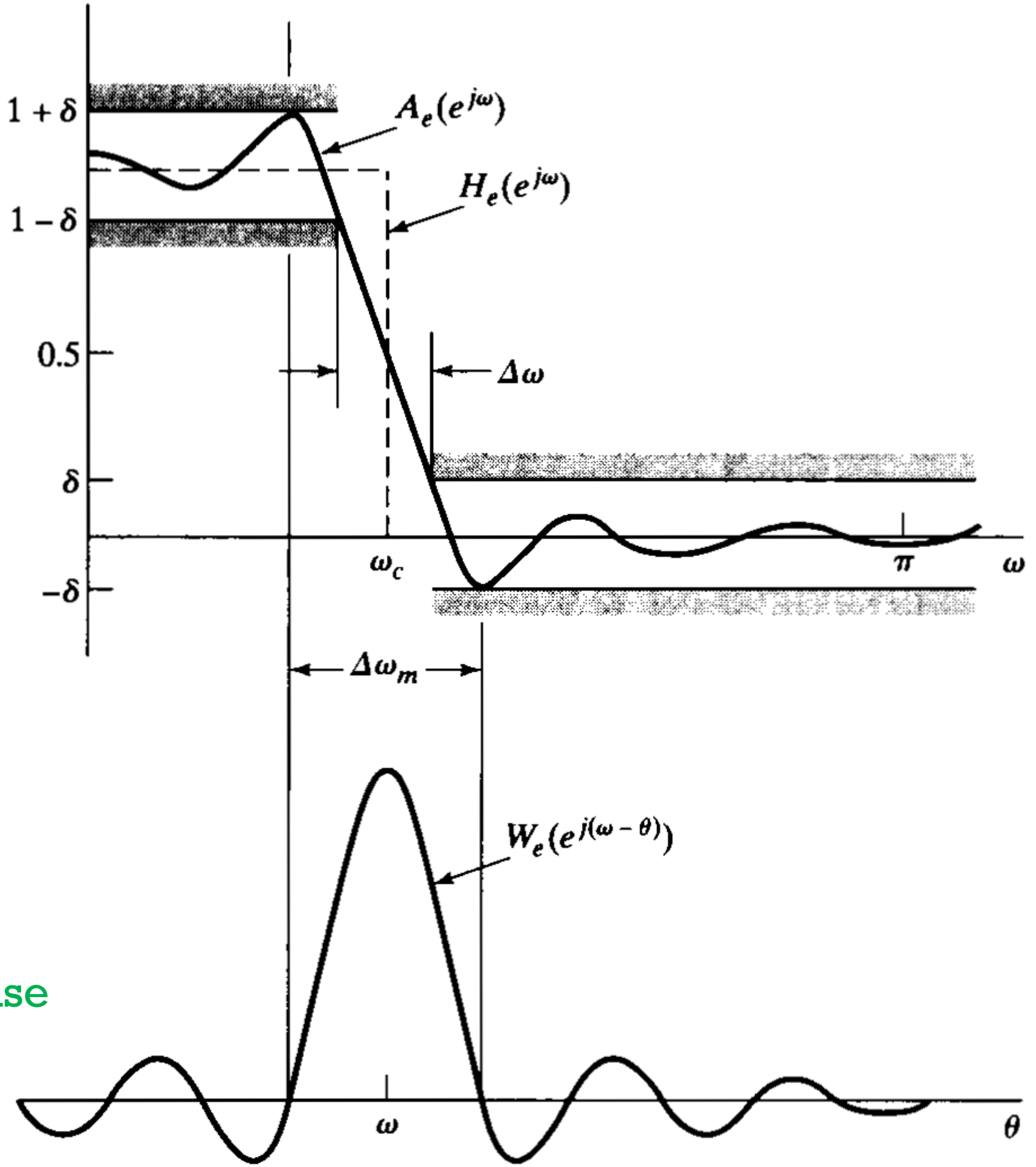
EXAMPLE

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases}$$

$$h_{\mathrm{lp}}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} e^{j\omega n}\, d\omega = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}$$

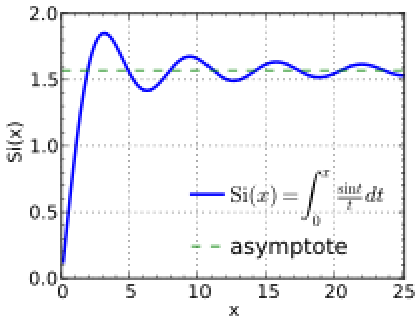
$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n],$$

symmetric
window to achieve linear –phase response



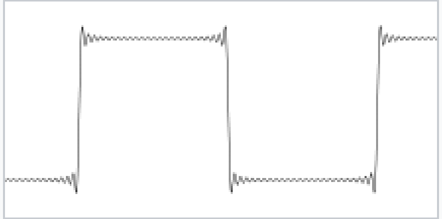
WHAT IS GIBBS'S PHENOMENON

The sine integral function, which gives the overshoot associated with the Gibbs phenomenon for the Fourier series of a step function on the real line

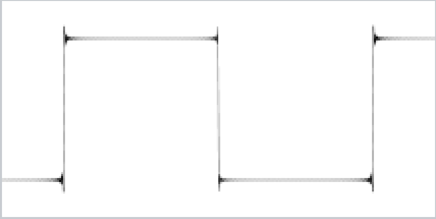


- The Gibbs phenomenon was first noticed and analyzed by Henry Wilbraham in an 1848 paper
- In 1898, Albert A. Michelson developed a device that could compute and re-synthesize the Fourier series. A widespread myth says that when the Fourier coefficients for a square wave were input to the machine, the graph would oscillate at the discontinuities, and that because it was a physical device subject to manufacturing flaws, Michelson was convinced that the overshoot was caused by errors in the machine. In fact, the graphs produced by the machine were not good enough to exhibit the Gibbs phenomenon clearly.
- Inspired by correspondence in Nature between Michelson and A. E. H. Love about the convergence of the Fourier series of the square wave function, J. Willard **Gibbs** published a note in 1898 pointing out the important distinction between the limit of the graphs of the partial sums of the Fourier series of a sawtooth wave and the graph of the limit of those partial sums. In his first letter Gibbs failed to notice the Gibbs phenomenon, and the limit that he described for the graphs of the partial sums was inaccurate. In 1899 he published a correction in which he described the overshoot at the point of discontinuity (Nature, April 27, 1899, p. 606). In 1906, Maxime Bôcher gave a detailed mathematical analysis of that overshoot, coining the term "**Gibbs phenomenon**" and bringing the term into widespread use.
- After the existence of Henry Wilbraham's paper became widely known, in 1925 Horatio Scott Carslaw remarked, "We may still call this property of Fourier's series (and certain other series) Gibbs's phenomenon; but we must no longer claim that the property was first discovered by Gibbs."

Functional approximation of square wave using 5 harmonics

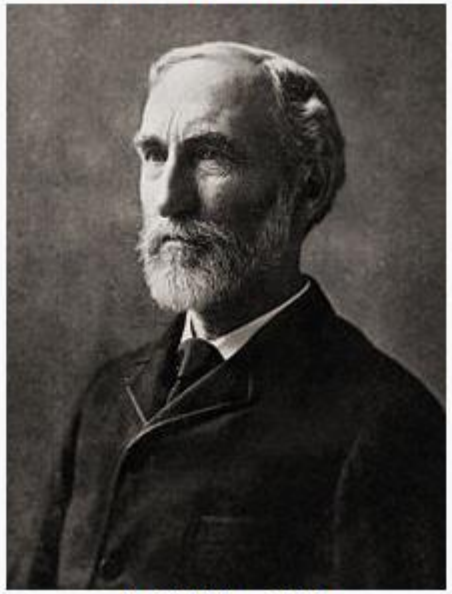


Functional approximation of square wave using 25 harmonics



Functional approximation of square wave using 125 harmonics

Josiah Willard Gibbs



Josiah Willard Gibbs

Born	February 11, 1839 New Haven, Connecticut, U.S.
Died	April 28, 1903 (aged 64) New Haven, Connecticut, U.S.
Nationality	American
Alma mater	Yale College
Known for	List [show]
Awards	Rumford Prize (1880) ForMemRS (1897)[1] Copley Medal (1901)
Scientific career	
Fields	Chemistry · Mathematics · Physics

PHASE DISCONTINUITY

$$H^z(\tau) = 1 - z^{-1}$$
$$H^f(\theta) = 1 - e^{-j\theta}$$
$$= 2\sin\left(\frac{\theta}{2}\right)e^{j\frac{1}{2}(\pi-\theta)}$$

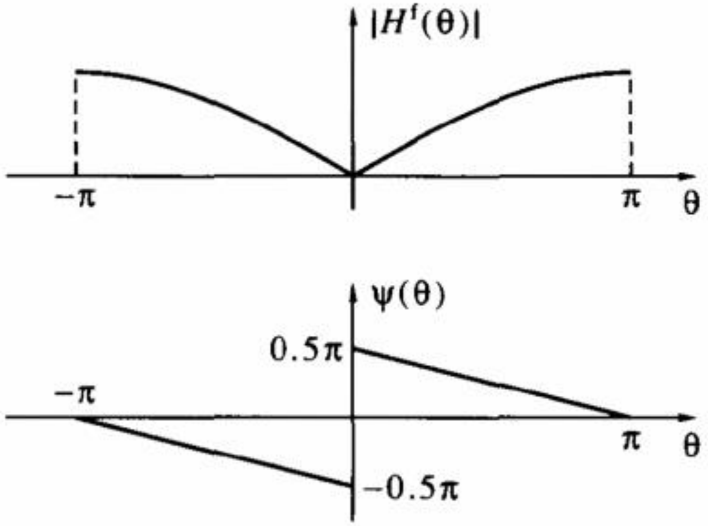


Figure 8.8 The magnitude and phase responses of $H^f(\theta)$

(piecewise phase)
segmented phase

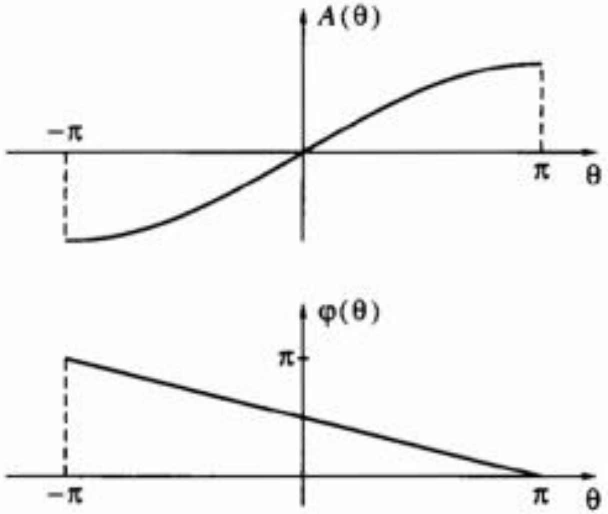


Figure 8.9 The amplitude and continuous-phase responses of $H^f(\theta)$

continuous phase

IMPLEMENTATION IN REAL-TIME

- Relation of filters realizations in real-time.
- We've seen that all the realizations are based on the collection of additional, multiplications and delays.
- The general realization in DSP is:

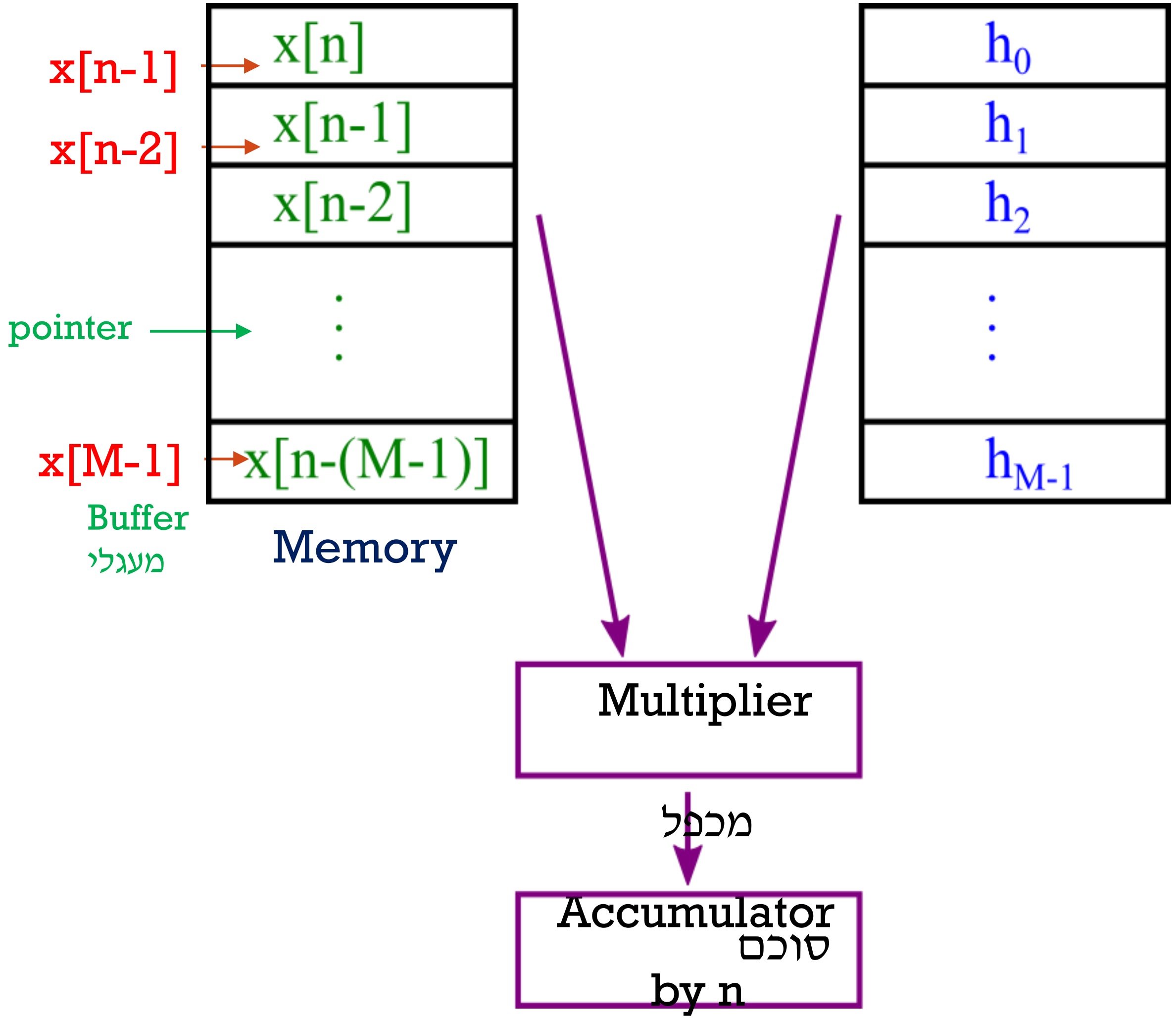
$$y[n] = \sum_{m=0}^{M-1} h_m x[n-m]$$

- The pointer is pointing on the current location in the circular array and this way one can address $x[n] \dots x[n-M+1]$

IMPLEMENTATION IN REAL-TIME

- After M multiplications and additions, we receive the current $y[n]$ for the FIR.

- Realization by command Assembler MAC- **M**ultiply and **AC**cumulate. כולל מיעון מעגלי



EXAMPLE 1
FIR

- Let assume a digital filter with impulse response as:

$$h[n] = \left[\frac{1}{2}\right]^n u[n] \qquad n = -\infty, .., \infty$$

- We will calculate the frequency response of the filter

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} \overrightarrow{h[n]}e^{-j\theta n} = \sum_{n=0}^{\infty} \left[\frac{1}{2}\right]^n e^{-j\theta n}$$

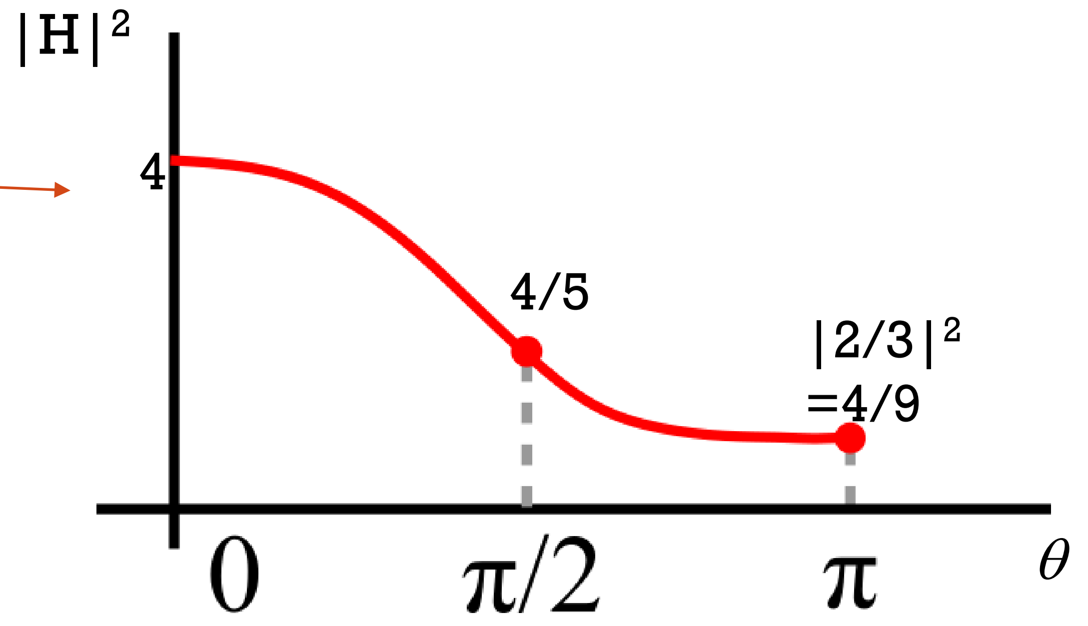
$$= \sum_{n=0}^{\infty} \left[\frac{1}{2}e^{-j\theta}\right]^n$$

Geometrical progression

$$= \frac{1}{1 - \frac{1}{2}e^{-j\theta}}$$

EXAMPLE 1: SOLUTION – FREQUENCY RESPONSE

- Calculate $|H|^2$ in $\theta = \left\{0, \frac{\pi}{2}, \pi\right\}$
- The filter will look like:



$$H(e^{j\theta}) = \frac{1}{1 - \frac{1}{2}e^{-j\theta}}$$

- 1) What is the filter type?

Answer: LPF

- 2) Can this filter be implemented by FIR concept

Answer: No. The infinite response can be approximated since $h[n]$ decays with time.

EXAMPLE 2
IIR

- Let analyze the following filter:

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

IIR

- Now we will calculate the frequency response:

$$Y(e^{j\theta}) = X(e^{j\theta}) + \frac{1}{2} Y(e^{j\theta}) e^{-j\theta}$$

$$Y(e^{j\theta}) \left[1 - \frac{1}{2} e^{-j\theta}\right] = X(e^{j\theta})$$

$$H(e^{j\theta}) = \frac{Y(e^{j\theta})}{X(e^{j\theta})} = \frac{1}{1 - \frac{1}{2} e^{-j\theta}}$$

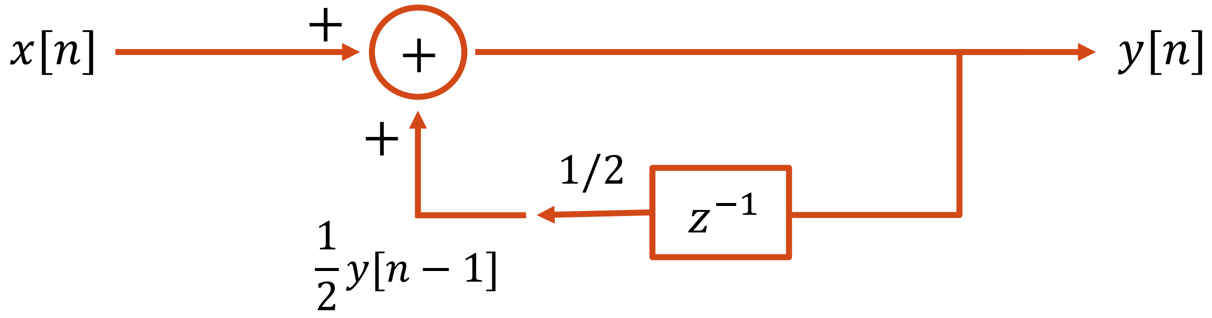
- This is identical to the DTFT of the filter in Example 1.

EXAMPLE 2: SOLUTION

- In FIR representation of the impulse response, one needs ∞ coefficients.
- In IIR representation one needs only three coefficients: $b = \{1\}, a = \{1, -1/2\}$

  y[n]   y[n-1]

- One can implement the filter that with FIR cannot be implemented.



$$\boldsymbol{y[n] = x[n] + \frac{1}{2}y[n-1]}$$

EXAMPLE 3

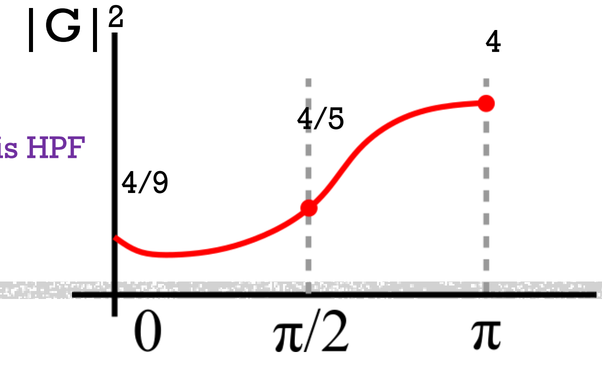
- We create a new filter via the multiplication in time $g[n] = h[n][-1]^n$
- What is the influence on the frequency response?

<u>Solution</u>:

$$h[n] = \left[\frac{1}{2}\right]^n u[n]$$

$$g[n] = \left[-\frac{1}{2}\right]^n u[n]$$

Then: $G(e^{j\theta}) = \frac{1}{1+\frac{1}{2}e^{-j\theta}}$



EXAMPLE 3: SOLUTION

- In general:

$$g[n] = h[n](-1)^n = h[n]e^{j\pi n}$$

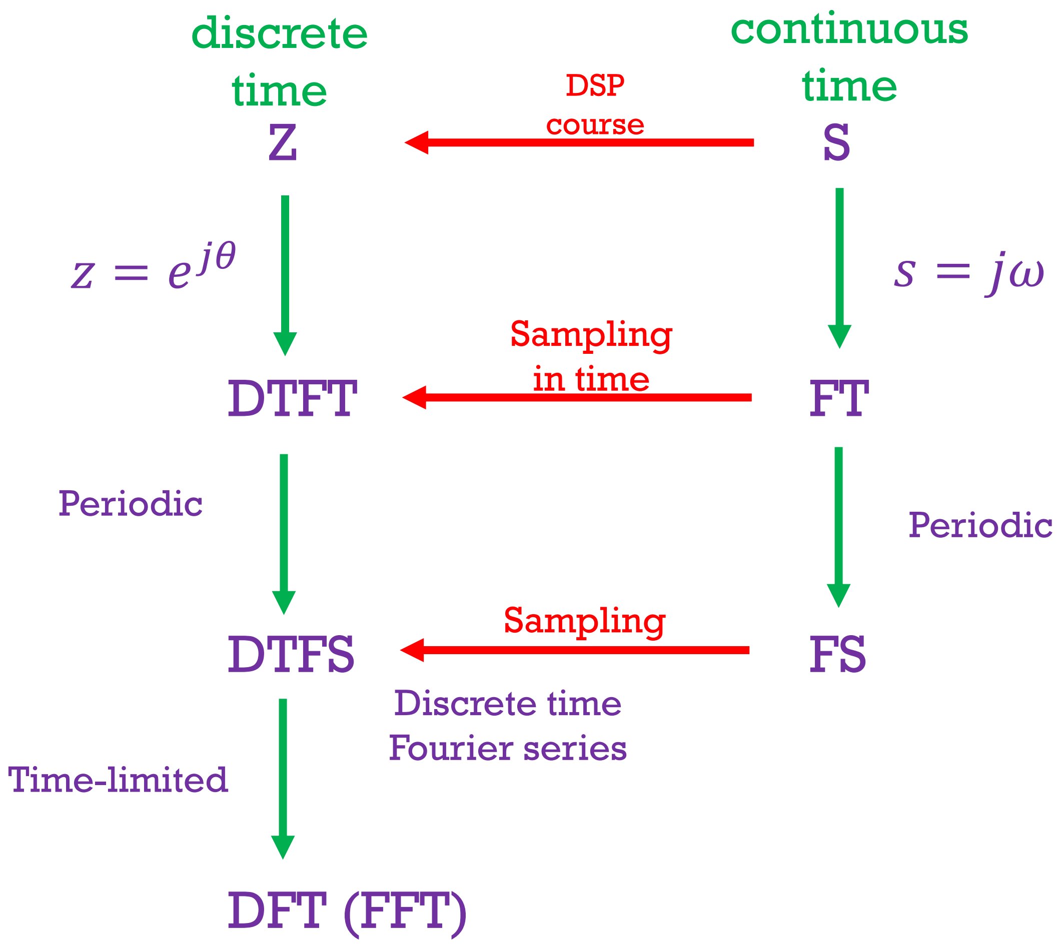
- And in frequency domain:

$$G(e^{j\theta}) = H(e^{j\theta}) * 2\pi\delta(\theta - \pi)$$

- We obtain the shift in π, therefore LPF to HPF and vise versa

RELATION BETWEEN THE TRANSFORMS

- During the course we've studied 7 different transforms



Sampling

Impulse train	$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad P(j\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \quad \omega_s = \frac{2\pi}{T}$
Ideal sampling	$x_s(t) = x(t)p(t)$
FT of sampled signal	$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$
Reconstruction (Shannon)	$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t-nT}{T}\right)$
Reconstruction 0-order	$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\Pi\left(\frac{t-nT-T/2}{T}\right)$
Reconstruction 1st-order	$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\Lambda\left(\frac{t-nT}{T}\right)$
Frequency relations	$\theta = \omega T = 2\pi \frac{f}{f_s}$

LIST OF EQUATIONS

- Sampling

Discrete Fourier transform (DFT)

Definition	$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk}$	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}$
Time shifting	$x[((n-n_0))_N]$	$e^{-j\frac{2\pi}{N}n_0 k}X[k]$
Time reversal	$x^*[((-n))_N]$	$X^*[k]$
Conjugation	$x^*[n]$	$X^*[((-k))_N]$
Symmetry	$x[n]$ real	$X[k] = X^*[((-k))_N]$
Convolution	$\sum_{m=0}^{N-1} x[m]y[((n-m))_N]$	$X[k]Y[k]$
Multiplication	$x[n]y[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} X[l]Y[((k-l))_N]$
Parseval	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 \quad =$	$\frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$
Matrix	$\mathbf{X} = \mathbf{F}\mathbf{x}, \quad \mathbf{F}^H\mathbf{F} = N\mathbf{I}$	

LIST OF EQUATIONS

- DFT

Fast Fourier transform (FFT)

Radix-2	$X[k] = F_1[k] + W_N^{-k} F_2[k], \quad X[k+\frac{N}{2}] = F_1[k] - W_N^{-k} F_2[k]$
	$0 \le k \le \frac{N}{2} - 1$, $\frac{N}{2} \log_2 N$ multiplications
Cooley-Tukey	$X[p,q] = \sum_{l=0}^{L-1} W_N^{-ql} \left[\sum_{m=0}^{M-1} x[l,m] W_M^{-mq} \right] W_L^{-lp}$
	$0 \le p \le L-1, \quad 0 \le q \le M-1$, $LM(L+M+1)$ multiplications

Filters

FIR	$y[n] = \sum_{l=0}^{L-1} b_l x[n-l]$
IIR	$y[n] = \sum_{l=0}^{L-1} b_l x[n-l] - \sum_{m=1}^{M-1} a_m y[n-m]$
Symmetric FIR	$h[n] = \pm h[M-1-n]$

LIST OF EQUATIONS

- FFT, Filters

LIST OF EQUATIONS

- FS

Fourier series (FS)

Definition	$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$
Time shifting	$x(t-\tau)$	$e^{-jk\omega_0\tau}a_k$
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(at), a>0$ (periodic $\frac{T}{a}$)	a_k
Conjugation	$x^*(t)$	a^*_{-k}
Symmetry	$x(t)$ real	$a_k=a^*_{-k}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(t)dt,\ a_0=0$	$\frac{a_k}{jk\omega_0}$
Convolution	$\int_T x(\tau)\,y(t-\tau)d\tau$	Ta_kb_k
Multiplication	$x(t)y(t)$	$\sum_{m=-\infty}^{\infty} a_m b_{k-m}$
Cosine	$2A\cos(\omega_0 t+B)$	$a_1=Ae^{jB}, a_{-1}=Ae^{-jB}$
Parseval	$\frac{1}{T}\int_T \lvert x(t)\rvert^2dt \quad =$	$\sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$

Fourier transform (FT)

Definition	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
Time shifting	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X(j\omega/a)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Symmetry	$x(t)$ real	$X(j\omega) = X^*(-j\omega)$
Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(ju)Y(j\omega-ju)du$
Delta	$\delta(t)$	1
One	1	$2\pi\delta(\omega)$
Exponent	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
Cosine	$\cos(w_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
Sine	$\sin(w_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
Unit step	$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
Decaying step	$u(t)e^{-at}$, $a > 0$	$\frac{1}{a+j\omega}$
Rectangular pulse	$\Pi(\frac{t}{T})$	$T\text{sinc}(\frac{\omega}{2\pi/T})$
Sinc	$\text{sinc}(\frac{t}{T})$	$T\,\Pi(\frac{\omega}{2\pi/T})$
Parseval	$\int_{-\infty}^{\infty} \|x(t)\|^2 dt \quad =$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(j\omega)\|^2 d\omega$

LIST OF EQUATIONS

- FT

Discrete-time Fourier transform (DTFT)

Definition	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})e^{j\theta n}d\theta$	$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$
Time shifting	$x[n-n_0]$	$e^{-j\theta n_0}X(e^{j\theta})$
Time reversal	$x[-n]$	$X(e^{-j\theta})$
Conjugation	$x^*[n]$	$X^*(e^{-j\theta})$
Symmetry	$x[n]$ real	$X(e^{j\theta}) = X^*(e^{-j\theta})$
Convolution	$\sum_{m=-\infty}^{\infty} x[m]y[n-m]$	$X(e^{j\theta})Y(e^{j\theta})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\phi})Y(e^{j(\theta-\phi)})d\phi$
Delta	$\delta[n]$	1
One	1	$2\pi\sum_{m=-\infty}^{\infty}\delta(\theta-2\pi m)$
Exponent	$e^{j\theta_0 n}$	$2\pi\sum_{m=-\infty}^{\infty}\delta(\theta-\theta_0-2\pi m)$
Cosine	$\cos[\theta_0 n]$	$\pi\sum_{m=-\infty}^{\infty}[\delta(\theta-\theta_0-2\pi m)+\delta(\theta+\theta_0-2\pi m)]$
Sine	$\sin[\theta_0 n]$	$\frac{\pi}{j}\sum_{m=-\infty}^{\infty}[\delta(\theta-\theta_0-2\pi m)-\delta(\theta+\theta_0-2\pi m)]$
Decaying step	$u[n]a^n, \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\theta}}$
Rectangular pulse	$\Pi_N[n]$	$\frac{\sin[\theta(N+\frac{1}{2})]}{\sin(\theta/2)}$
Sinc (normalized)	$\frac{\sin[Wn]}{\pi n}$	$\sum_{m=-\infty}^{\infty}\Pi(\frac{\theta-2\pi m}{2W})$
Parseval	$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2 \quad =$	$\frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\theta})\rvert^2 d\theta$

LIST OF EQUATIONS

DTFT

Z transform

Definition	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$	$X(z)$	$\sum_{n=-\infty}^{\infty} x[n] z^{-n}$
Time shifting	$x[n - n_0]$		$z^{-n_0} X(z)$
time reversal	$x[-n]$		$X(z^{-1})$
Conjugation	$x^*[n]$		$X^*(z^*)$
Convolution	$\sum_{m=-\infty}^{\infty} x[m] y[n-m]$		$X(z)Y(z)$
Delta	$\delta[n]$		1
Decaying step	$a^n u[n]$		$\frac{1}{1-az^{-1}} \quad (\lvert z \rvert > a)$

LIST OF EQUATIONS

- Z transform

General

Rectangular pulse, $x \in \mathrm{R}$	$\Pi(x) = \{1,\ \lvert x\rvert < \frac{1}{2};\quad \frac{1}{2},\ \lvert x\rvert = \frac{1}{2};\quad 0,\ \text{elsewhere}\}$
Rectangular pulse, $n \in \mathrm{Z}$	$\Pi_N[n] = \{1,\ \lvert n\rvert \leq N;\quad 0,\ \text{elsewhere}\}$
Triangle function, $x \in \mathrm{R}$	$\Lambda(x) = \{1 - \lvert x\rvert,\ \lvert x\rvert < 1;\quad 0,\ \text{elsewhere}\}$
Unit step, $x \in \mathrm{R}$	$u(x) = \{1,\ x > 0;\quad \frac{1}{2},\ x = 0;\quad 0,\ \text{elsewhere}\}$
Unit step, $n \in \mathrm{Z}$	$u[n] = \{1,\ n \geq 0;\quad 0,\ \text{elsewhere}\}$
Sinc, $x \in \mathrm{R}$	$\mathrm{sinc}(\mathrm{x}) = \frac{\sin(\pi \mathrm{x})}{\pi \mathrm{x}}$
Sine-sine product	$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$
Cosine-cosine product	$\cos(a)\cos(b) = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$
Cosine-sine product	$\cos(a)\sin(b) = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$
Eigen-decomposition	$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$
Numbers	Real: $\mathbb{R}$, Complex: $\mathbb{C}$, Integer: $\mathbb{Z}$, Natural: $\mathbb{N}$

LIST OF EQUATIONS

General relations