

DISCRETE FOURIER TRANSFORM (DFT): MORE MATRIX FORMULATIONS, CIRCULAR CONVOLUTION

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Reading: Chapter 4 by B. Porat

DFT IN MATRIX FORMULATION

$$\begin{array}{c} N \times 1 \\ \left[\begin{array}{c} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{array} \right] \end{array} = \underbrace{\begin{array}{c} N \times N \\ \left[\begin{array}{ccccc} W_N^0 & W_N^0 & W_N^0 & \dots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \dots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \dots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \dots & W_N^{-(N-1)^2} \end{array} \right] \end{array}}_{\text{DFT Matrix}} \times \begin{array}{c} N \times 1 \\ \left[\begin{array}{c} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{array} \right] \end{array}$$

$k=0$

DFT \rightarrow 37C

IDFT: MATRIX FORMULATION

From IDFT definition: $\underline{x} = \frac{1}{N} \underline{F}^* \underline{X}$
 Inverse matrix DFT $= \underline{F}^{-1} \underline{X}$

הפוך DFT
 כ"כ מציגים

הכיוון של DFT

1. שורה! המשוואה המלאה
 2. המשוואה סטטית כי המילים שורה k ומהמקום המדויק ה.
 3. המשוואה המלאה
- Hermitian transpose $\underline{F}^H = \underline{F}^T$
 פז ארנס פז פז
- $\underline{F}^H = \underline{F}^T = \underline{F}$
- $\underline{F}^H = \underline{F}^T = \underline{F}$
- ה"כ מציגים

n, m
 סדר הריבוי n, m

$$\sum_{k=0}^{n-1} W_n^{-kn} \cdot W_n^{km} = n \cdot \delta(n-m) \rightarrow$$

$$\frac{1}{\sqrt{n}} \cdot F =$$

$$\left(\frac{1}{\sqrt{n}} F \right)^k \cdot \left(\frac{1}{\sqrt{n}} \underline{F} \right) = \underline{1}$$

$(F_n)_{km} = W_n^{-km}$

בסדר זה:

$X_1 - X_2$
 $X_1 - X_2$

DFT MATRIX: PROPERTIES

The DFT matrix has the following properties:

1. The elements on the first row and the first column are 1, since $W_N^0 = 1$.
2. It is symmetric, since its (k, n) th element, W_N^{-kn} , is symmetric in k and n . $\underline{\underline{F^T = F}}$
3. If \bar{F}_N is the complex conjugate of the DFT matrix, then

Note, in Matlab X' is Hermite and transpose

$$F_N^H \bar{F}_N' = N I_N, \quad (4.23)$$

where I_N is the $N \times N$ identity matrix. This follows from (4.6), since

$$(F_N \bar{F}_N')_{k,l} = \sum_{n=0}^{N-1} W_N^{-kn} W_N^{ln} = \sum_{n=0}^{N-1} W_N^{(l-k)n} = N \delta[(k-l) \bmod N]. \quad (4.24)$$

A complex $N \times N$ matrix Q satisfying $Q\bar{Q}' = I_N$ is called a *unitary matrix*. If Q is real, it is called an *orthonormal matrix*. It follows from (4.23) that the matrix $N^{-1/2}F_N$ is unitary. It is also symmetric (property 2), so it is a *symmetric unitary matrix*. The matrix $N^{-1/2}F_N$ is called the *normalized DFT matrix*.

BASIS VECTORS REPRESENTATION

Each set of N complex orthonormal vectors $x[n]$, $X[k]$ of length N belongs to the vectorial space of rank N

$$\underline{x}, \underline{X} \in \mathbb{C}^N$$

For instance, real \underline{x} is named *Natural basis*, *Standard basis*, *Canonical basis*

This basis is composed of $\delta[n - m]$ for each $0 \leq m \leq N - 1$. So the overall set of vectorial basis is I .

$$x[n] = \sum_{m=0}^{N-1} \delta[n - m] x[m]$$

m: 0 *m: 1* *m: 2* *m: 3* *m: 4* *m: 5* *m: 6* *m: 7* *m: 8* *m: 9* *m: 10* *m: 11* *m: 12* *m: 13* *m: 14* *m: 15* *m: 16* *m: 17* *m: 18* *m: 19* *m: 20* *m: 21* *m: 22* *m: 23* *m: 24* *m: 25* *m: 26* *m: 27* *m: 28* *m: 29* *m: 30* *m: 31* *m: 32* *m: 33* *m: 34* *m: 35* *m: 36* *m: 37* *m: 38* *m: 39* *m: 40* *m: 41* *m: 42* *m: 43* *m: 44* *m: 45* *m: 46* *m: 47* *m: 48* *m: 49* *m: 50* *m: 51* *m: 52* *m: 53* *m: 54* *m: 55* *m: 56* *m: 57* *m: 58* *m: 59* *m: 60* *m: 61* *m: 62* *m: 63* *m: 64* *m: 65* *m: 66* *m: 67* *m: 68* *m: 69* *m: 70* *m: 71* *m: 72* *m: 73* *m: 74* *m: 75* *m: 76* *m: 77* *m: 78* *m: 79* *m: 80* *m: 81* *m: 82* *m: 83* *m: 84* *m: 85* *m: 86* *m: 87* *m: 88* *m: 89* *m: 90* *m: 91* *m: 92* *m: 93* *m: 94* *m: 95* *m: 96* *m: 97* *m: 98* *m: 99* *m: 100* *m: 101* *m: 102* *m: 103* *m: 104* *m: 105* *m: 106* *m: 107* *m: 108* *m: 109* *m: 110* *m: 111* *m: 112* *m: 113* *m: 114* *m: 115* *m: 116* *m: 117* *m: 118* *m: 119* *m: 120* *m: 121* *m: 122* *m: 123* *m: 124* *m: 125* *m: 126* *m: 127* *m: 128* *m: 129* *m: 130* *m: 131* *m: 132* *m: 133* *m: 134* *m: 135* *m: 136* *m: 137* *m: 138* *m: 139* *m: 140* *m: 141* *m: 142* *m: 143* *m: 144* *m: 145* *m: 146* *m: 147* *m: 148* *m: 149* *m: 150* *m: 151* *m: 152* *m: 153* *m: 154* *m: 155* *m: 156* *m: 157* *m: 158* *m: 159* *m: 160* *m: 161* *m: 162* *m: 163* *m: 164* *m: 165* *m: 166* *m: 167* *m: 168* *m: 169* *m: 170* *m: 171* *m: 172* *m: 173* *m: 174* *m: 175* *m: 176* *m: 177* *m: 178* *m: 179* *m: 180* *m: 181* *m: 182* *m: 183* *m: 184* *m: 185* *m: 186* *m: 187* *m: 188* *m: 189* *m: 190* *m: 191* *m: 192* *m: 193* *m: 194* *m: 195* *m: 196* *m: 197* *m: 198* *m: 199* *m: 200* *m: 201* *m: 202* *m: 203* *m: 204* *m: 205* *m: 206* *m: 207* *m: 208* *m: 209* *m: 210* *m: 211* *m: 212* *m: 213* *m: 214* *m: 215* *m: 216* *m: 217* *m: 218* *m: 219* *m: 220* *m: 221* *m: 222* *m: 223* *m: 224* *m: 225* *m: 226* *m: 227* *m: 228* *m: 229* *m: 230* *m: 231* *m: 232* *m: 233* *m: 234* *m: 235* *m: 236* *m: 237* *m: 238* *m: 239* *m: 240* *m: 241* *m: 242* *m: 243* *m: 244* *m: 245* *m: 246* *m: 247* *m: 248* *m: 249* *m: 250* *m: 251* *m: 252* *m: 253* *m: 254* *m: 255* *m: 256* *m: 257* *m: 258* *m: 259* *m: 260* *m: 261* *m: 262* *m: 263* *m: 264* *m: 265* *m: 266* *m: 267* *m: 268* *m: 269* *m: 270* *m: 271* *m: 272* *m: 273* *m: 274* *m: 275* *m: 276* *m: 277* *m: 278* *m: 279* *m: 280* *m: 281* *m: 282* *m: 283* *m: 284* *m: 285* *m: 286* *m: 287* *m: 288* *m: 289* *m: 290* *m: 291* *m: 292* *m: 293* *m: 294* *m: 295* *m: 296* *m: 297* *m: 298* *m: 299* *m: 300* *m: 301* *m: 302* *m: 303* *m: 304* *m: 305* *m: 306* *m: 307* *m: 308* *m: 309* *m: 310* *m: 311* *m: 312* *m: 313* *m: 314* *m: 315* *m: 316* *m: 317* *m: 318* *m: 319* *m: 320* *m: 321* *m: 322* *m: 323* *m: 324* *m: 325* *m: 326* *m: 327* *m: 328* *m: 329* *m: 330* *m: 331* *m: 332* *m: 333* *m: 334* *m: 335* *m: 336* *m: 337* *m: 338* *m: 339* *m: 340* *m: 341* *m: 342* *m: 343* *m: 344* *m: 345* *m: 346* *m: 347* *m: 348* *m: 349* *m: 350* *m: 351* *m: 352* *m: 353* *m: 354* *m: 355* *m: 356* *m: 357* *m: 358* *m: 359* *m: 360* *m: 361* *m: 362* *m: 363* *m: 364* *m: 365* *m: 366* *m: 367* *m: 368* *m: 369* *m: 370* *m: 371* *m: 372* *m: 373* *m: 374* *m: 375* *m: 376* *m: 377* *m: 378* *m: 379* *m: 380* *m: 381* *m: 382* *m: 383* *m: 384* *m: 385* *m: 386* *m: 387* *m: 388* *m: 389* *m: 390* *m: 391* *m: 392* *m: 393* *m: 394* *m: 395* *m: 396* *m: 397* *m: 398* *m: 399* *m: 400* *m: 401* *m: 402* *m: 403* *m: 404* *m: 405* *m: 406* *m: 407* *m: 408* *m: 409* *m: 410* *m: 411* *m: 412* *m: 413* *m: 414* *m: 415* *m: 416* *m: 417* *m: 418* *m: 419* *m: 420* *m: 421* *m: 422* *m: 423* *m: 424* *m: 425* *m: 426* *m: 427* *m: 428* *m: 429* *m: 430* *m: 431* *m: 432* *m: 433* *m: 434* *m: 435* *m: 436* *m: 437* *m: 438* *m: 439* *m: 440* *m: 441* *m: 442* *m: 443* *m: 444* *m: 445* *m: 446* *m: 447* *m: 448* *m: 449* *m: 450* *m: 451* *m: 452* *m: 453* *m: 454* *m: 455* *m: 456* *m: 457* *m: 458* *m: 459* *m: 460* *m: 461* *m: 462* *m: 463* *m: 464* *m: 465* *m: 466* *m: 467* *m: 468* *m: 469* *m: 470* *m: 471* *m: 472* *m: 473* *m: 474* *m: 475* *m: 476* *m: 477* *m: 478* *m: 479* *m: 480* *m: 481* *m: 482* *m: 483* *m: 484* *m: 485* *m: 486* *m: 487* *m: 488* *m: 489* *m: 490* *m: 491* *m: 492* *m: 493* *m: 494* *m: 495* *m: 496* *m: 497* *m: 498* *m: 499* *m: 500* *m: 501* *m: 502* *m: 503* *m: 504* *m: 505* *m: 506* *m: 507* *m: 508* *m: 509* *m: 510* *m: 511* *m: 512* *m: 513* *m: 514* *m: 515* *m: 516* *m: 517* *m: 518* *m: 519* *m: 520* *m: 521* *m: 522* *m: 523* *m: 524* *m: 525* *m: 526* *m: 527* *m: 528* *m: 529* *m: 530* *m: 531* *m: 532* *m: 533* *m: 534* *m: 535* *m: 536* *m: 537* *m: 538* *m: 539* *m: 540* *m: 541* *m: 542* *m: 543* *m: 544* *m: 545* *m: 546* *m: 547* *m: 548* *m: 549* *m: 550* *m: 551* *m: 552* *m: 553* *m: 554* *m: 555* *m: 556* *m: 557* *m: 558* *m: 559* *m: 560* *m: 561* *m: 562* *m: 563* *m: 564* *m: 565* *m: 566* *m: 567* *m: 568* *m: 569* *m: 570* *m: 571* *m: 572* *m: 573* *m: 574* *m: 575* *m: 576* *m: 577* *m: 578* *m: 579* *m: 580* *m: 581* *m: 582* *m: 583* *m: 584* *m: 585* *m: 586* *m: 587* *m: 588* *m: 589* *m: 590* *m: 591* *m: 592* *m: 593* *m: 594* *m: 595* *m: 596* *m: 597* *m: 598* *m: 599* *m: 600* *m: 601* *m: 602* *m: 603* *m: 604* *m: 605* *m: 606* *m: 607* *m: 608* *m: 609* *m: 610* *m: 611* *m: 612* *m: 613* *m: 614* *m: 615* *m: 616* *m: 617* *m: 618* *m: 619* *m: 620* *m: 621* *m: 622* *m: 623* *m: 624* *m: 625* *m: 626* *m: 627* *m: 628* *m: 629* *m: 630* *m: 631* *m: 632* *m: 633* *m: 634* *m: 635* *m: 636* *m: 637* *m: 638* *m: 639* *m: 640* *m: 641* *m: 642* *m: 643* *m: 644* *m: 645* *m: 646* *m: 647* *m: 648* *m: 649* *m: 650* *m: 651* *m: 652* *m: 653* *m: 654* *m: 655* *m: 656* *m: 657* *m: 658* *m: 659* *m: 660* *m: 661* *m: 662* *m: 663* *m: 664* *m: 665* *m: 666* *m: 667* *m: 668* *m: 669* *m: 670* *m: 671* *m: 672* *m: 673* *m: 674* *m: 675* *m: 676* *m: 677* *m: 678* *m: 679* *m: 680* *m: 681* *m: 682* *m: 683* *m: 684* *m: 685* *m: 686* *m: 687* *m: 688* *m: 689* *m: 690* *m: 691* *m: 692* *m: 693* *m: 694* *m: 695* *m: 696* *m: 697* *m: 698* *m: 699* *m: 700* *m: 701* *m: 702* *m: 703* *m: 704* *m: 705* *m: 706* *m: 707* *m: 708* *m: 709* *m: 710* *m: 711* *m: 712* *m: 713* *m: 714* *m: 715* *m: 716* *m: 717* *m: 718* *m: 719* *m: 720* *m: 721* *m: 722* *m: 723* *m: 724* *m: 725* *m: 726* *m: 727* *m: 728* *m: 729* *m: 730* *m: 731* *m: 732* *m: 733* *m: 734* *m: 735* *m: 736* *m: 737* *m: 738* *m: 739* *m: 740* *m: 741* *m: 742* *m: 743* *m: 744* *m: 745* *m: 746* *m: 747* *m: 748* *m: 749* *m: 750* *m: 751* *m: 752* *m: 753* *m: 754* *m: 755* *m: 756* *m: 757* *m: 758* *m: 759* *m: 760* *m: 761* *m: 762* *m: 763* *m: 764* *m: 765* *m: 766* *m: 767* *m: 768* *m: 769* *m: 770* *m: 771* *m: 772* *m: 773* *m: 774* *m: 775* *m: 776* *m: 777* *m: 778* *m: 779* *m: 780* *m: 781* *m: 782* *m: 783* *m: 784* *m: 785* *m: 786* *m: 787* *m: 788* *m: 789* *m: 790* *m: 791* *m: 792* *m: 793* *m: 794* *m: 795* *m: 796* *m: 797* *m: 798* *m: 799* *m: 800* *m: 801* *m: 802* *m: 803* *m: 804* *m: 805* *m: 806* *m: 807* *m: 808* *m: 809* *m: 810* *m: 811* *m: 812* *m: 813* *m: 814* *m: 815* *m: 816* *m: 817* *m: 818* *m: 819* *m: 820* *m: 821* *m: 822* *m: 823* *m: 824* *m: 825* *m: 826* *m: 827* *m: 828* *m: 829* *m: 830* *m: 831* *m: 832* *m: 833* *m: 834* *m: 835* *m: 836* *m: 837* *m: 838* *m: 839* *m: 840* *m: 841* *m: 842* *m: 843* *m: 844* *m: 845* *m: 846* *m: 847* *m: 848* *m: 849* *m: 850* *m: 851* *m: 852* *m: 853* *m: 854* *m: 855* *m: 856* *m: 857* *m: 858* *m: 859* *m: 860* *m: 861* *m: 862* *m: 863* *m: 864* *m: 865* *m: 866* *m: 867* *m: 868* *m: 869* *m: 870* *m: 871* *m: 872* *m: 873* *m: 874* *m: 875* *m: 876* *m: 877* *m: 878* *m: 879* *m: 880* *m: 881* *m: 882* *m: 883* *m: 884* *m: 885* *m: 886* *m: 887* *m: 888* *m: 889* *m: 890* *m: 891* *m: 892* *m: 893* *m: 894* *m: 895* *m: 896* *m: 897* *m: 898* *m: 899* *m: 900* *m: 901* *m: 902* *m: 903* *m: 904* *m: 905* *m: 906* *m: 907* *m: 908* *m: 909* *m: 910* *m: 911* *m: 912* *m: 913* *m: 914* *m: 915* *m: 916* *m: 917* *m: 918* *m: 919* *m: 920* *m: 921* *m: 922* *m: 923* *m: 924* *m: 925* *m: 926* *m: 927* *m: 928* *m: 929* *m: 930* *m: 931* *m: 932* *m: 933* *m: 934* *m: 935* *m: 936* *m: 937* *m: 938* *m: 939* *m: 940* *m: 941* *m: 942* *m: 943* *m: 944* *m: 945* *m: 946* *m: 947* *m: 948* *m: 949* *m: 950* *m: 951* *m: 952* *m: 953* *m: 954* *m: 955* *m: 956* *m: 957* *m: 958* *m: 959* *m: 960* *m: 961* *m: 962* *m: 963* *m: 964* *m: 965* *m: 966* *m: 967* *m: 968* *m: 969* *m: 970* *m: 971* *m: 972* *m: 973* *m: 974* *m: 975* *m: 976* *m: 977* *m: 978* *m: 979* *m: 980* *m: 981* *m: 982* *m: 983* *m: 984* *m: 985* *m: 986* *m: 987* *m: 988* *m: 989* *m: 990* *m: 991* *m: 992* *m: 993* *m: 994* *m: 995* *m: 996* *m: 997* *m: 998* *m: 999* *m: 1000*

We will write

$$\underline{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x[0] + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} x[1] + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} x[N-1]$$

coefficients

basis vectors
וקטורי הבסיס

BASIS VECTORS REPRESENTATION

One can represent \underline{x} (in-time) via **DFT basis**

Transpose and Hermite

$$\underline{x} = \frac{1}{N} \underline{F}^H \underline{X} = \frac{1}{N} \underline{F}^* \underline{X}$$

We will write:

$$\underline{x} = \frac{1}{N} X[0] \underline{F}_0^* + \frac{1}{N} X[1] \underline{F}_1^* + \dots + \frac{1}{N} X[N-1] \underline{F}_{N-1}^*$$

while: $\underline{F}^* = [\underline{F}_0^*, \underline{F}_1^*, \dots, \underline{F}_{N-1}^*]$

Columns of
DFT matrix

1st
column 2nd
column

with \underline{F}^* inverse (complex conjugate) DFT matrix

\underline{F}_i^* - the columns are **basis vectors**

BASIS VECTOR REPRESENTATION

Handwritten notes in Hebrew: x is spanned by the basis, meaning that it is given by the linear combination of $\frac{1}{N} F^*$, while the coefficients are $X[k]$ which are \underline{X} .

\underline{x} is spanned by the basis, meaning that it is given by the linear combination of $\frac{1}{N} \underline{F}^*$, while

the coefficients are $\underline{X}[k]$ which are \underline{X} .

- ✓ \underline{F}_i^* are the basis because we've got N orthogonal vectors.
- ✓ $\underline{X}[k]$ are the projection coefficients of $\underline{x}[n]$ on the space that is spanned by \underline{F}^* .

EXAMPLE: COS

$$x[n] = \cos[\theta_0 n], \quad n = 0, \dots, N-1 \quad \theta_0 = \frac{2\pi m}{N}$$

therefore $0 \leq n \leq N-1$ $x[n] = \frac{1}{2} e^{j \frac{2\pi}{N} m n} + \frac{1}{2} e^{j \frac{2\pi}{N} [N-m] n}$

Euler

f_m f_{N-m}

the exponents are two columns in DFT basis F_m^*, F_{N-m}^*



Therefore, in this case one can represent it by two basis vectors

SIMILAR TO CONTINUOUS TIME: COMPLEX REPRESENTATIONS OF REAL SIGNALS

$$\checkmark \quad x(t) = A \cos(\omega_0 t + \theta) = \frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)} = \frac{1}{2} \tilde{x}(t) + \frac{1}{2} \tilde{x}^*(t)$$

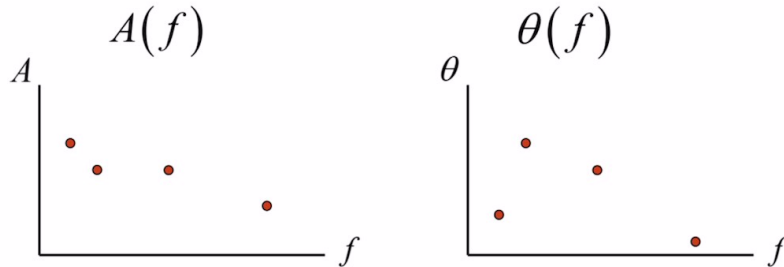
- We think of a real sinusoid as a sum of two **phasors**
 - One moving forward in time
 - One moving backward in time

SIMILAR TO CONTINUOUS TIME: FREQUENCY REPRESENTATIONS

- Any sum of sines and cosines can be reduced to a **canonical** form

$$A_1 \cos(\omega_1 + \theta_1) + A_2 \cos(\omega_2 + \theta_2) + \cdots + A_n \cos(\omega_n + \theta_n)$$

So, we can define the amplitude spectrum and phase

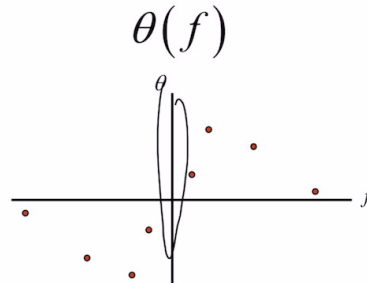
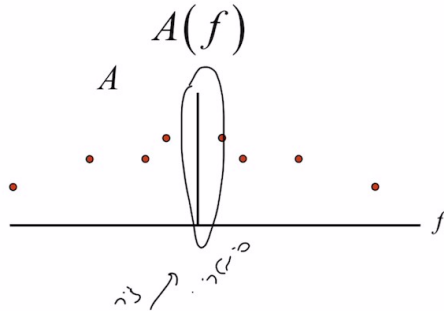


COMPLEX FREQUENCY REPRESENTATION

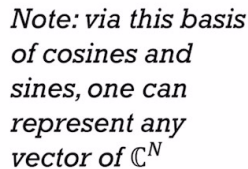
- Another canonical form is complex ✓

$$\frac{A_1}{2} e^{-j(\omega_1 + \theta_1)} + \frac{A_1}{2} e^{j(\omega_1 + \theta_1)} + \dots + \frac{A_n}{2} e^{-j(\omega_n + \theta_n)} + \frac{A_n}{2} e^{j(\omega_n + \theta_n)}$$

So, we define the two-sided spectra



$\cos x = \sin$



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EXAMPLE: A DFT THAT CAN BE COMPUTED ANALYTICALLY

- Calculate N=8 points DFT of

$$x[n] = \underset{\swarrow}{\textcircled{4}} + 3 \sin \left[\frac{\pi n}{2} \right] \nwarrow$$

$$\underline{\underline{N = 8}}$$

EXAMPLE: A DFT THAT CAN BE COMPUTED ANALYTICALLY

- We 'simplify' by expanding the trigonometric function into exponential form
- Then we plug into the definition of the DFT

$$\swarrow$$

$$x[n] = 4 + 3 \sin \left[\frac{\pi n}{2} \right] = 4 - j \frac{3}{2} e^{j \frac{\pi n}{2}} + j \frac{3}{2} e^{-j \frac{\pi n}{2}} \quad N = 8$$

$$\swarrow$$

$$X[k] = \sum_{n=0}^7 x[n] (W_8)^{-kn}$$

$$= 4 \sum_{n=0}^7 e^{-j \frac{2\pi kn}{8}} - j \frac{3}{2} \sum_{n=0}^7 e^{j \frac{\pi n}{2}} e^{-j \frac{2\pi kn}{8}} + j \frac{3}{2} \sum_{n=0}^7 e^{-j \frac{\pi n}{2}} e^{-j \frac{2\pi kn}{8}}$$

$$X[k] = 4 \sum_{n=0}^7 e^{-j\frac{2\pi kn}{8}} - j\frac{3}{2} \sum_{n=0}^7 e^{j\frac{\pi n}{2}} e^{-j\frac{2\pi kn}{8}} + j\frac{3}{2} \sum_{n=0}^7 e^{-j\frac{\pi n}{2}} e^{-j\frac{2\pi kn}{8}}$$



$$X[k] = 4 \sum_{n=0}^7 (W_8)^{-kn} - j\frac{3}{2} \sum_{n=0}^7 (W_8)^{(2-k)n} + j\frac{3}{2} \sum_{n=0}^7 (W_8)^{-(2+k)n}$$

$$= 4 \cdot 8\delta[0] - j\frac{3}{2} \cdot 8\delta[2] + j\frac{3}{2} \cdot 8\delta[6]$$

$$= 32\delta[0] - j12\delta[2] + j12\delta[6]$$

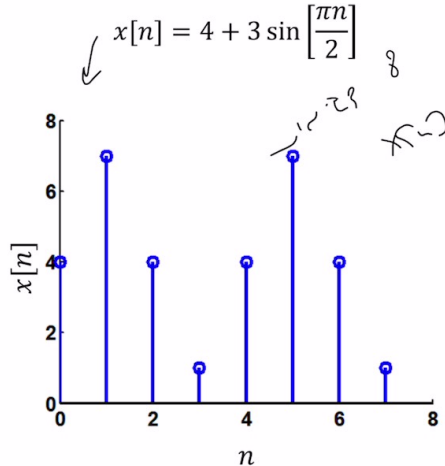
7

THE RESULTS OF THE ANALYTIC COMPUTATION

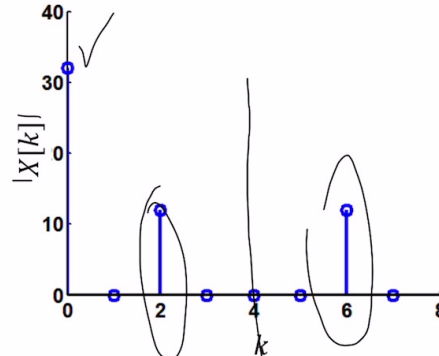
- What about symmetry properties?
- Which frequencies does x have?

Symmetry property: $W_N^{k+N/2} = -W_N^k$

Periodicity property: $W_N^{k+N} = W_N^k$



$$X[k] = 32\delta[0] - j12\delta[2] + j12\delta[6]$$



SURVEY: DFT MATRIX PROPERTIES WITH LINEAR ALGEBRA ONLY



EasyPolls:

Computing x from X using inverse (F) is possible:

☒ For all finite X

☐ Only for real and finite X

☐ For all finite X except $[0,0,\dots,0]$

results

vote

בדיקה של G עם 2^N
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SURVEY: DFT MATRIX PROPERTIES WITH LINEAR ALGEBRA ONLY



EasyPolls:

הקצרה
באורך $N=4$ נכתב x
הערך באמצעות
המשתנים המאזנה
הקובעים מה ההתאמה

- $N=4$ ✓
- $x=[1,2,3,4]$ $y=\text{IDFT}\{X \cdot \exp(-j2\pi k/4)\}=?$
- ☐ [0,1,2,3]
- ☒ [4,1,2,3]
- ☐ [2,3,4,0]
- ☐ [2,3,4,1]
- $N=4$ ✓
- $W = e^{-j2\pi k/N}$

results

vote

CIRCULAR SHIFT SURVEY SOLUTION

הזזה מעגלית

$$x[n] = \{1, 2, 3, 4\}$$

$n = 0 \quad 1 \quad 2 \quad 3$

$$y[n] = \left[((n-1))_4 \right] = \{4, 1, 2, 3\}$$

$n = 0 \quad 1 \quad 2 \quad 3$

Handwritten notes and mappings:

- Blue dashed line: $n-1 \Rightarrow$
- Yellow boxes highlight the values in the second sequence: $x[3]=4$, $x[0]=1$, $x[1]=2$, $x[2]=3$.
- Purple arrows show the mapping from the first sequence to the second: $x[0] \rightarrow y[3]$, $x[1] \rightarrow y[0]$, $x[2] \rightarrow y[1]$, $x[3] \rightarrow y[2]$.
- Handwritten note: $n=0$ with an arrow pointing to the first row of the second sequence.

Now we will prove:

$$y[n] = x \left[((n - m))_N \right] \overset{\text{DFT}}{\leftrightarrow} X[k] W_N^{-km}$$

CIRCULAR SHIFT PROPERTY: PROOF

$$x[((n-m))_N] \overset{\text{DFT}}{\leftrightarrow} X[k]W_N^{-km}$$

- Let assume $y[n] = x[((n-m))_N]$ and calculate $Y[k]$

$$\begin{aligned} Y[k] &= \sum_{n=0}^{N-1} x[((n-m))_N] W_N^{-kn} = \sum_{n=0}^{N-1} x[((n-m))_N] W_N^{-k(n-m+m)} \\ &= W_N^{-km} \sum_{n=0}^{N-1} x[((n-m))_N] W_N^{-k(n-m)} = W_N^{-km} \sum_{n=0}^{N-1} x[((n-m))_N] W_N^{-k((n-m))_N} \\ &= W_N^{-km} X[k] \end{aligned}$$

Periodic in N

The sum, is like DFT
but the order is different

SHIFT IN FREQUENCY

For given $x[n] \overset{\text{DFT}}{\leftrightarrow} X[k]$

and $x[n] W_N^{ln} \leftrightarrow X[(k - l)_N]$?

- Proof: H.W.

Handwritten notes in purple ink: W_N^{ln} and $k - l$.

MATRIX REPRESENTATION: PROPERTIES

1. Linearity

$$z[n] = ax[n] + by[n] \iff Z^d[k] = aX^d[k] + bY^d[k], \quad a, b \in \mathbb{C}.$$

The proof follows immediately from (4.3).

2. Periodicity

$$X^d[k] = X^d[k + N].$$

3. Circular shift

$$y[n] = x[(n - m) \bmod N] \iff Y^d[k] = W_N^{-km} X^d[k], \quad m \in \mathbb{Z}. \quad (4.30)$$

Example: $l-k$ is in range $-(N-1), \dots, 0, \dots, N-1$

Proof

$$D[(l-k) \bmod N] = \begin{cases} 1, & l-k = \text{multiplication of } N \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} Y^d[k] &= \sum_{n=0}^{N-1} x[(n - m) \bmod N] W_N^{-kn} = \sum_{n=0}^{N-1} x[(n - m) \bmod N] W_N^{-k(n-m+m)} \\ &= W_N^{-km} \sum_{n=0}^{N-1} x[(n - m) \bmod N] W_N^{-k(n-m) \bmod N} = W_N^{-km} X^d[k]. \quad (4.31) \end{aligned}$$

In passing from the first to the second line we used the property

$$W_N^{-k(l \bmod N)} = W_N^{-kl}, \quad \text{for all } k, l \in \mathbb{Z}.$$

MATRIX REPRESENTATION OF CIRCULAR SHIFT

- Let assume $x[n]$ with length of N , $0 \leq n \leq N - 1$. The signal is shifted in a circular way in m samples to obtain $y[n]$. $y[n] = x \left[((n - m))_N \right]$, $0 \leq n \leq N - 1$

Represent \underline{y} as \underline{x} via matrix representation

$$\underline{y} = \underline{A} \underline{x}$$

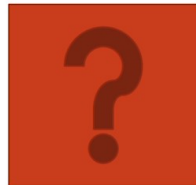
Handwritten notes in purple ink:

$y[n] = x[(n-m)_N]$
 $\underline{y} = \underline{A} \underline{x}$

Handwritten notes in purple ink:

$y[n] = x[(n-m)_N]$
 $\underline{y} = \underline{A} \underline{x}$

SURVEY: DFT MATRIX PROPERTIES WITH MATRIX NOTATION



- EasyPolls:

$y = Ax$ represents $y[n] = x[(n-1) \bmod 3]$. What is A ?

☐ [0 0 1; 0 1 0; 1 0 0]

☐ [0 1 0; 0 0 1; 1 0 0]

✓ ☒ [0 0 1; 1 0 0; 0 1 0]

results

vote



MATRIX REPRESENTATION OF CIRCULAR CONVOLUTION

- Let assume $x[n]$ with length of N , $0 \leq n \leq N-1$. The signal is shifted in a circular way in m samples to obtain $y[n]$. $y[n] = x[(n-m)_N]$, $0 \leq n \leq N-1$

Represent y as x via matrix representation

We will write $y[n] = \sum_{l=0}^{N-1} \delta[l - ((n-m)_N)] x[l]$ and in matrix representation

$$\underline{y} = \begin{bmatrix} \delta[l - ((0-m)_N)] \\ \delta[l - ((1-m)_N)] \\ \vdots \\ \delta[l - ((N-1-m)_N)] \end{bmatrix} \underline{x}$$

For instance, for $m=1, N=4$

$$\underline{y} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x} \rightarrow \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}^T \underline{x}$$

What is the inverse action of this? **1.** Shift to the left by one location; **2.** We can make three more identical shifts $\underline{A} \underline{A} \underline{A} = \text{inverse}$. \underline{A}^T since \underline{A} is orthogonal can be cascaded as $y = \underline{A}_1 \underline{A}_2 \underline{A}_3 x$;

EXERCISE: MATRIX NOTATION IN FREQUENCY

- Present circular shift by matrix representation in frequency.
- Let $y[n] = x[((n - m))_N]$

Find the relation in the frequency domain $\underline{Y} = \underline{B} \cdot \underline{X}$

$$Y[k] = W_N^{-km} X[k]$$

$$\underline{Y} = \underbrace{\begin{bmatrix} W_N^0 & & 0 \\ & \ddots & \\ 0 & & W_N^{-(N-1)m} \end{bmatrix}}_{\underline{B}} \underline{X}$$

$$\underline{B} \text{ diagonal} - \underline{B} = \text{diag}(W_N^{-km})$$

$$\text{diag}(W_N^{-km})$$

EXERCISE

I) in-time

$$\underline{y} = \underline{A} \underline{x}$$

we will calculate DFT:

$$\underline{F} \underline{y} = \underline{F} \underline{A} \frac{1}{N} \underline{F}^H \underline{X}$$

IDFT

$$\underline{F} \underline{y} = \left[\frac{1}{N} \underline{F} \underline{A} \underline{F}^H \right] \underline{X}$$

$$\underline{Y} = \underline{B} \underline{X}$$

diag(W_N^{-km})

II) in-frequency

H.W.

Prove that \underline{B} is the same for two cases



April 27, 1755 – August 16, 1836

PARSEVAL THEOREM

- It originates from a 1799 theorem about **series** by **Marc-Antoine Parseval**, which was later applied to the **Fourier series**. It is also known as **Rayleigh's energy theorem**, or **Rayleigh's identity**, after **John William Strutt**, Lord Rayleigh.

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]Y^*[k]$$

Special case:

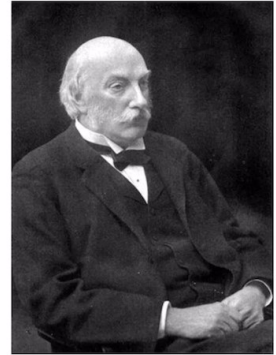
$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

PARSEVAL

- Marc-Antoine Parseval des Chênes (27 April 1755 – 16 August 1836) was a French mathematician, most famous for what is now known as Parseval's theorem, which presaged the unitarity of the Fourier transform.
- **Scientific career:** mathematics



LORD RAILEIGH



- John William Strutt, 3rd Baron Rayleigh, (12 November 1842 – 30 June 1919) was an English mathematician who made extensive contributions to science. He spent all of his academic career at the University of Cambridge.
- 1904 Laureate of the **Nobel Prize in Physics** "for his investigations of the densities of the most important gases and for his discovery of argon in connection with these studies."
- **Scientific career:** physics, optics, acoustics
- Rayleigh provided the first theoretical treatment of the **elastic scattering of light by particles much smaller than the light's wavelength**, a phenomenon now known as "Rayleigh scattering", which notably explains [why the sky is blue](#).