DISCRETE FOURIER TRANSFORM (DFT): MORE MATRIX FORMULATIONS, CIRCULAR CONVOLUTION

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Reading: Chapter 4 by B. Porat

DFT IN MATRIX FORMULATION

$k = 0$

$$
Nx1 \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \overset{NxN}{\underbrace{\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}}_{\text{DFT} \to \text{[illegible]}}} \times \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} Nx1
$$

IDFT: MATRIX FORMULATION

From IDFT definition: $\underline{x} = \frac{1}{N} \underline{\underline{F}}^{*} \underline{X}$

Inverse matrix DFT $= \underline{\underline{F}}^{-1} \underline{X}$

IDFT לפי ההגדרה בכתיב מטריציוני

תכונות מטריצת DFT:

1. שורה / עמודה ראשונות מכילות רק אחדים.
2. המטריצה סימטרית כי האיברים W_N^{-kn} תלויים באותו אופן באינדקס שורה k ובאינדקס העמודה n. ⇐

$\boxed{\underline{\underline{F}}^{T} = \underline{\underline{F}}}$

3. המטריצה מקיימת

Hermitian transpose

גם טרנספוז גם צמוד

$\underline{\underline{F}}^{H} \cdot \underline{\underline{F}} = N \cdot \underline{\underline{I}}$ ← מטריצת יחידה

כי $\sum_{k=0}^{N-1} W_N^{-kn} \cdot W_N^{km} = N \cdot \delta(n-m)$

איבר m,n של התוצאה

$\frac{1}{\sqrt{N}} \cdot \underline{\underline{F}}$ מטריצה יוניטרית

$$\left(\frac{1}{\sqrt{N}} \underline{\underline{F}}\right)^H \cdot \left(\frac{1}{\sqrt{N}} \underline{\underline{F}}\right) = \underline{\underline{I}}$$

$(F_N)_{k,m} = W_N^{-km}$

<u>בתרגול</u> :

X' - צמוד הרמיטי וגם טרנספוז

$X.'$ - רק טרנספוז

DFT MATRIX: PROPERTIES

The DFT matrix has the following properties:

1. The elements on the first row and the first column are 1, since $W_N^0 = 1$.
2. It is symmetric, since its (k,n)th element, W_N^{-kn}, is symmetric in k and n. $\underline{\underline{F}}^T = \underline{\underline{F}}$
3. If $\bar{\boldsymbol{F}}_N$ is the complex conjugate of the DFT matrix, then

Note, in Matlab X' is Hermite and transpose

$$\boldsymbol{F}_N^{\mathrm{H}} \bar{\boldsymbol{F}}'_N = N\boldsymbol{I}_N, \tag{4.23}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix. This follows from (4.6), since

$$(\boldsymbol{F}_N \bar{\boldsymbol{F}}'_N)_{k,l} = \sum_{n=0}^{N-1} W_N^{-kn} W_N^{ln} = \sum_{n=0}^{N-1} W_N^{(l-k)n} = N\delta[(k-l) \bmod N]. \tag{4.24}$$

A complex $N \times N$ matrix $\boldsymbol{Q}$ satisfying $\boldsymbol{Q}\bar{\boldsymbol{Q}}' = \boldsymbol{I}_N$ is called a *unitary matrix*. If Q is real, it is called an *orthonormal matrix*. It follows from (4.23) that the matrix $N^{-1/2}\boldsymbol{F}_N$ is unitary. It is also symmetric (property 2), so it is a *symmetric unitary matrix*. The matrix $N^{-1/2}\boldsymbol{F}_N$ is called the *normalized DFT matrix*.

BASIS VECTORS REPRESENTATION

Each set of N complex orthonormal vectors $x[\mathrm{n}]$, $X[\mathrm{k}]$ of length N belongs to the vectorial space of rank N

$$\underline{x}, \underline{X} \in \mathbb{C}^N$$

טבעי סטנדרטי קנוני

For instance, real $\underline{x}$ is named *Natural basis, Standard basis, Canonical basis*

This basis is composed of $\delta[n-m]$ for each $0 \leq m \leq N-1$. So the overall set of vectorial basis is $\boldsymbol{I}$.

$$x[\mathrm{n}] = \sum_{m=0}^{N-1} \delta[n-m]\, x[m]$$

מקדם

magnitude of δ

m=0

We will write

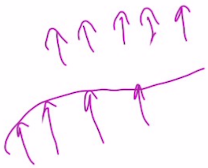
$$\underline{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x[0] + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x[1] + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x[N-1]$$

coefficients

m=1 m=N-1

basis vectors

וקטורי הבסיס



BASIS VECTORS REPRESENTATION

One can represent $\underline{x}$ (in-time) via DFT basis

Transpose and Hermite

IDFT

$$\underline{x} = \frac{1}{N}\underline{\underline{F}}^H\underline{X} = \frac{1}{N}\underline{\underline{F}}^*\underline{X} =$$

We will write: $\underline{x} = \frac{1}{N}X[0]\underline{F}_0^* + \frac{1}{N}X[1]\underline{F}_1^* + \ldots + \frac{1}{N}X[N-1]\underline{F}_{N-1}^*$

X(t)

while: $\underline{\underline{F}}^* = \left[\underline{F}_0^*, \underline{F}_1^*, \ldots, \underline{F}_{\mathrm{N}-1}^*\right]$

= DFT

Columns of DFT matrix

1st column 2nd column

$\underline{\underline{F}}$ DFT

$\underline{\underline{F}}^*$ IDFT

הופכית צמוד קומפלקסי

with $\underline{\underline{F}}^*$ inverse (complex conjugate) DFT matrix

$\underline{F}_{\mathrm{i}}^*$ - the columns are *basis vectors*

BASIS VECTOR REPRESENTATION

x נפרש ע"י הבסיס, כלומר נתון ע"י קומבינציה לינארית
של $\frac{1}{N}F_i^*$, וכאשר המקדמים הם X(k) כלומר $\underline{X}$.

$\underline{x}$ is spanned by the basis, meaning that it is given by the linear combination of $\frac{1}{N}\underline{\underline{F}}^*$, while the coefficients are $X[k]$ which are $\underline{X}$.

F_i^* מהווים בסיס כי הם וקטורים אורתוגונליים! (יש N כאלה).

- $\underline{F}_i^*$ are the basis because we've got N *orthogonal* vectors.

X(k) הם מקדמי ההטלה של x(n) על מרחב שנפרש ע"י $\underline{\underline{F}}^*$.

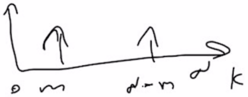
- $X[k]$ are the projection coefficients of $x[n]$ on the space that is spanned by $\underline{\underline{F}}^*$.

EXAMPLE: COS

$$x[n] = \cos[\theta_0 n],\ n = 0,\ldots,N-1 \qquad \theta_0 = \frac{2\pi m}{N}$$

therefore $\quad 0 \le n \le N-1 \quad x[n] \overset{\text{Euler}}{=} \underbrace{\frac{1}{2} e^{j\frac{2\pi}{N}mn}}_{\mathbf{f}_m} + \underbrace{\frac{1}{2} e^{j\frac{2\pi}{N}[N-m]n}}_{\mathbf{f}_{N-m}}$

the exponents are two columns in DFT basis $F^*_{\mathrm{m}}, F^*_{N-\mathrm{m}}$



Therefore, in this case one can represent it by two basis vectors

SIMILAR TO CONTINUOUS TIME: COMPLEX REPRESENTATIONS OF REAL SIGNALS

$$x(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2}e^{j(\omega_0 t+\theta)} + \frac{A}{2}e^{-j(\omega_0 t+\theta)} = \tfrac{1}{2}\tilde{x}(t) + \tfrac{1}{2}\tilde{x}^*(t)$$

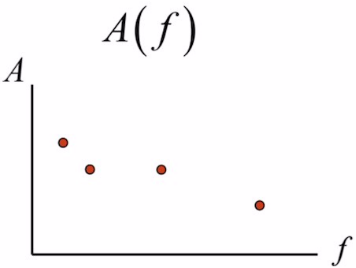
- We think of a real sinusoid as a sum of two **phasors**
  - One moving forward in time
  - One moving backward in time

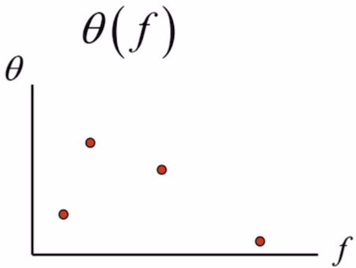
SIMILAR TO CONTINUOUS TIME: FREQUENCY REPRESENTATIONS

- Any sum of sines and cosines can be reduced to a **canonical** form

$$A_1 \cos(\omega_1 + \theta_1) + A_2 \cos(\omega_2 + \theta_2) + \cdots + A_n \cos(\omega_n + \theta_n)$$

So, we can define the amplitude spectrum and phase





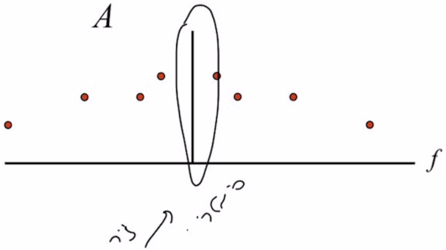
COMPLEX FREQUENCY REPRESENTATION

- Another canonical form is complex

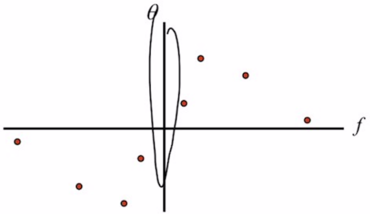
$$\frac{A_1}{2}e^{-j(\omega_1+\theta_1)}+\frac{A_1}{2}e^{j(\omega_1+\theta_1)}+\cdots+\frac{A_n}{2}e^{-j(\omega_n+\theta_n)}+\frac{A_n}{2}e^{j(\omega_n+\theta_n)}$$

So, we define the two-sided spectra

$A(f)$



$\theta(f)$



DEMONSTRATION OF DFT

cos + i sin

$N=8$

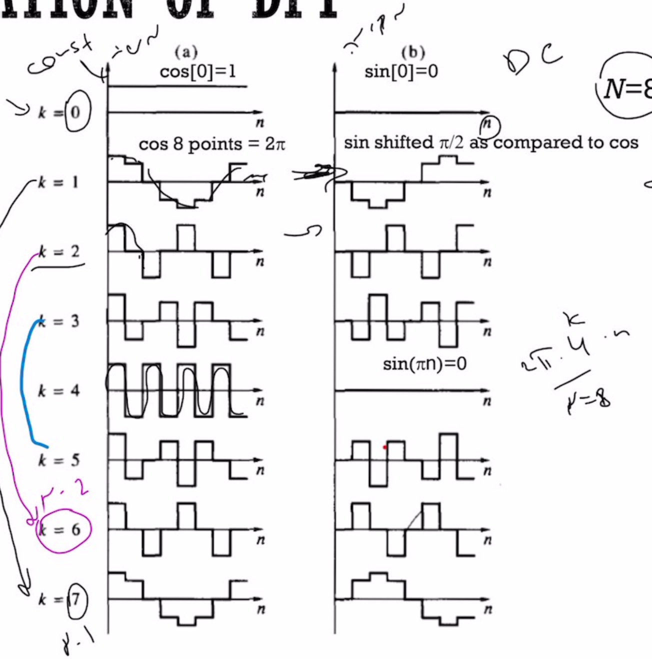
8 points = 2π

2 points = $\pi/2$

Symmetry in k=1 and k=7

Symmetry in k=2 and k=6

Symmetry in k=3 and k=5



Symmetry in k=3 and k=5

Symmetry in k=2 and k=6

Symmetry in k=1 and k=7

Figure 4.5 The DFT basis vectors for $N = 8$: (a) real part; (b) imaginary part.

Note: via this basis of cosines and sines, one can represent any vector of $\mathbb{C}^N$

EXAMPLE: A DFT THAT CAN BE COMPUTED ANALYTICALLY

- Calculate N=8 points DFT of

$$x[n] = 4 + 3\sin\left[\frac{\pi n}{2}\right]$$

$$N = 8$$

EXAMPLE: A DFT THAT CAN BE COMPUTED ANALYTICALLY

- We 'simplify' by expanding the trigonometric function into exponential form
- Then we plug into the definition of the DFT

$$x[n] = 4 + 3\sin\left[\frac{\pi n}{2}\right] = 4 - j\frac{3}{2}e^{j\frac{\pi n}{2}} + j\frac{3}{2}e^{-j\frac{\pi n}{2}} \qquad N = 8$$

$$X[k] = \sum_{n=0}^{7} x[n](W_8)^{-kn}$$

$$= 4\sum_{n=0}^{7} e^{-j\frac{-2\pi k n}{8}} - j\frac{3}{2}\sum_{n=0}^{7} e^{j\frac{\pi n}{2}}e^{-j\frac{2\pi k n}{8}} + j\frac{3}{2}\sum_{n=0}^{7} e^{-j\frac{\pi n}{2}}e^{-j\frac{2\pi k n}{8}}$$

$$X[k] = 4\sum_{n=0}^{7} e^{-j\frac{-2\pi kn}{8}} - j\frac{3}{2}\sum_{n=0}^{7} e^{j\frac{\pi n}{2}} e^{-j\frac{2\pi kn}{8}} + j\frac{3}{2}\sum_{n=0}^{7} e^{-j\frac{\pi n}{2}} e^{-j\frac{2\pi kn}{8}}$$

$$X[k] = 4\sum_{n=0}^{7} (W_8)^{-kn} - j\frac{3}{2}\sum_{n=0}^{7} (W_8)^{(2-k)n} + j\frac{3}{2}\sum_{n=0}^{7} (W_8)^{-(2+k)n}$$

$$= 4 \cdot 8\delta[0] - j\frac{3}{2} \cdot 8\delta[2] + j\frac{3}{2} \cdot 8\delta[6]$$

$$= 32\delta[0] - j12\delta[2] + j12\delta[6]$$

THE RESULTS OF THE ANALYTIC COMPUTATION

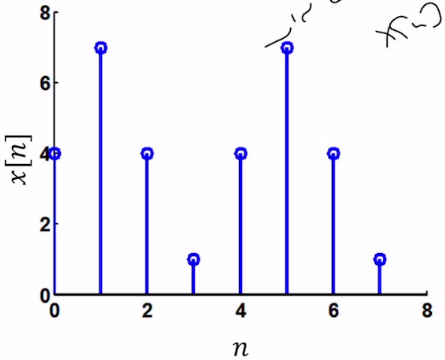
Symmetry property:	$W_N^{k+N/2} = -W_N^k$
Periodicity property:	$W_N^{k+N} = W_N^k$

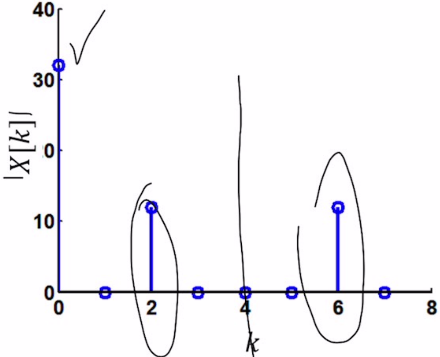
- What about symmetry properties?
- Which frequencies does x have?

$$x[n] = 4 + 3\sin\left[\frac{\pi n}{2}\right]$$

8

$$X[k] = 32\delta[0] - j12\delta[2] + j12\delta[6]$$





SURVEY: DFT MATRIX PROPERTIES WITH LINEAR ALGEBRA ONLY

- EasyPolls:

Computing x from X using inverse(F) is possible:

- ✓ ○ For all finite X
- × ○ Only for real and finite X
- × ○ For all finite X except [0,0,...,0]

results **vote**

SURVEY: DFT MATRIX PROPERTIES WITH LINEAR ALGEBRA ONLY

- EasyPolls:

N=4

x=[1,2,3,4], y=IDFT{X*exp(-j2*pi*k/4)}=?

N=4

- [0,1,2,3]
- [4,1,2,3]
- [2,3,4,0]
- [2,3,4,1]

results vote

CIRCULAR SHIFT הזזה מעגלית

- What is the meaning of a shift in finite sequences?
- By making linear shift we will loose information

- Let assume sequence $x[n]$ of length N
- The circular shift in m is $x[(n-m)\mathrm{mod}N]$ or $x[((n-m))_N]$
- We will show that $x[([n-m])_N] \overset{DFT}{\leftrightarrow} X[k]W_N^{-km}$

נניח סדרה $x[n]$ באורך N

$0 \le n \le N-1$.

כשהתמרת ה-DFT $X[k]$ באורך N.

CIRCULAR SHIFT SURVEY SOLUTION הזזה מעגלית

$x[n] = \{1, 2, 3, 4\}$

n = 0 1 2 3

$y[n] = \left[((n-1))_4\right] = \{4, 1, 2, 3\}$

n = 0 1 2 3

n

$\left[((0-1))_4\right] = x[3] = 4$

$\left[((1-1))_4\right] = x[0] = 1$

$\left[((2-1))_4\right] = x[1] = 2$

$\left[((3-1))_4\right] = x[2] = 3$

n=-1=

n-1=

4-1=3

n = 0

Now we will prove: $y[n] = x\left[((n-m))_N\right] \overset{\text{DFT}}{\leftrightarrow} X[k]W_N^{-km}$

CIRCULAR SHIFT PROPERTY: PROOF

$$x\left[((n-m))_N\right] \overset{\text{DFT}}{\leftrightarrow} X[k]W_N^{-km}$$

- Let assume $y[n] = x\left[((n-m))_N\right]$ and calculate $Y[k]$

$$Y[k] = \sum_{n=0}^{N-1} x\left[((n-m))_N\right] W_N^{-kn} = \sum_{n=0}^{N-1} x\left[((n-m))_N\right] W_N^{-k(n-m+m)}$$

$$= W_N^{-km} \sum_{n=0}^{N-1} x\left[((n-m))_N\right] W_N^{-k(n-m)} = W_N^{-km} \sum_{n=0}^{N-1} x\left[((n-m))_N\right] W_N^{-k((n-m))_N} =$$

Periodic in N

$$= W_N^{-km} X[k]$$

The sum, is like DFT
but the order is different

SHIFT IN FREQUENCY

For given $x[n] \overset{\text{DFT}}{\leftrightarrow} X[k]$

and $x[n]\,W_N^{ln} \leftrightarrow X[((n-m))_N]$

- Proof: H.W.

MATRIX REPRESENTATION: PROPERTIES

1. **Linearity**

$$z[n] = ax[n] + by[n] \iff Z^{\mathrm{d}}[k] = aX^{\mathrm{d}}[k] + bY^{\mathrm{d}}[k], \quad a, b \in \mathbb{C}.$$

The proof follows immediately from (4.3).

2. **Periodicity**

$$X^{\mathrm{d}}[k] = X^{\mathrm{d}}[k+N].$$

3. **Circular shift**

$$y[n] = x[(n-m) \bmod N] \iff Y^{\mathrm{d}}[k] = W_N^{-km} X^{\mathrm{d}}[k], \quad m \in \mathbb{Z}. \tag{4.30}$$

Example: l-k is in range –(N-1),….,0,…..,N-1

$$D[(l\text{-}k)\bmod N]=\begin{cases} 1, l-k = multiplication\ of\ N \\ 0, otherwise \end{cases}$$

Proof

$$Y^{\mathrm{d}}[k] = \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-kn} = \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-k(n-m+m)}$$

$$= W_N^{-km} \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-k[(n-m) \bmod N]} = W_N^{-km} X^{\mathrm{d}}[k]. \tag{4.31}$$

In passing from the first to the second line we used the property

$$W_N^{-k(l \bmod N)} = W_N^{-kl}, \quad \text{for all } k, l \in \mathbb{Z}.$$

MATRIX REPRESENTATION OF CIRCULAR SHIFT

- Let assume x[n] with length of N, $0 \le n \le N-1$. The signal is shifted in a circular way in m samples to obtain y[n]. $y[n] = x\left[\left((n-m)\right)_N\right], 0 \le n \le N-1$

יש לי x[n] באורך N, ואני נזיז
באופן מעגלי ? m צעדים
ונקבל y[n].

Represent $\underline{y}$ as $\underline{x}$ via matrix representation

$$\underline{y} = \underline{\underline{A}}\,\underline{x}$$

ייצג את ייצוג: כ הגדרה $\underline{y} = \underline{\underline{A}}\,\underline{x}$
מטריצי

SURVEY: DFT MATRIX PROPERTIES WITH MATRIX NOTATION

- EasyPolls:

y=Ax represents y[n]=x[((n-1))mod3. What is A?

- [0 0 1; 0 1 0; 1 0 0]
- [0 1 0; 0 0 1; 1 0 0]
- [0 0 1; 1 0 0; 0 1 0]

results vote

n=3

1 0 0
0 1 0
0 0 1

MATRIX REPRESENTATION OF CIRCULAR ~~CONVOLUTION~~ Shift

- Let assume x[n] with length of N, $0 \le n \le N-1$. The signal is shifted in a circular way in m samples to obtain y[n]. $y[n] = x\left[((n-m))_N\right], 0 \le n \le N-1$

Represent $\underline{y}$ as $\underline{x}$ via matrix representation

$$\underline{y} = \underline{\underline{A}}\,\underline{x}$$

We will write $y[n] = \sum_{l=0}^{N-1} \delta[l-((n-m))_N]x[l]$ and in matrix representation

$$\underline{y} = \begin{array}{c c} & l \rightarrow \\ n \downarrow & \begin{bmatrix} \delta[l-((0-m))_N] \\ \delta[l-((1-m))_N] \\ \vdots \\ \delta[l-((N-1-m))_N] \end{bmatrix} \end{array} \underline{x}$$

For instance, for m=1, N=4

$$\underline{y} = \overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underline{x} \rightarrow \overset{\underline{y}}{\begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \overset{\underline{x}}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}} \rightarrow \begin{matrix} [1\ 2\ 3\ 4]^T\ \underline{\underline{x}} \\ [4\ 1\ 2\ 3]^T\ \underline{y} \end{matrix}$$

What is the inverse action of this? **1.** Shift to the left by one location; **2.** We can make three more identical shifts $\underline{\underline{A}}\ \underline{\underline{A}}\ \underline{\underline{A}}$ =inverse. A^T since A is orthogonal can be cascaded as $y=A_1A_2A_3x$;

EXERCISE: MATRIX NOTATION IN FREQUENCY

- Present circular shift by matrix representation in frequency.
- Let $y[n] = x[((n - m))_N]$

תלות ב-m

Find the relation in the frequency domain $\underline{Y} = \underline{\underline{B}} \cdot \underline{X}$

$Y[k] = W_N^{-km} X[k]$

$$\underline{Y} = \underbrace{\begin{bmatrix} W_N^0 & & 0 \\ & \ddots & \\ 0 & & W_N^{-(N-1)m} \end{bmatrix}}_{\underline{\underline{B}}} \underline{X}$$

$\underline{\underline{B}}$ diagonal - $\underline{\underline{B}} = \mathrm{diag}\left(W_N^{-km}\right)$

$\mathrm{diag}\left(W_N^{-km}\right)$

מטריצה
אלכסונית

EXERCISE

I) in-time

$$\underline{y} = \underline{\underline{A}} \cdot \underline{x}$$

we will calculate DFT:

DFT

$$\underline{\underline{F}}\,\underline{y} = \underline{\underline{F}}\,\underline{\underline{A}}\underbrace{\frac{1}{N}\underline{\underline{F}}^{H}\underline{X}}_{\underline{x}}$$

IDFT

$$\underline{\underline{F}}\,\underline{y} = \left[\frac{1}{N}\underline{\underline{F}}\,\underline{\underline{A}}\,\underline{\underline{F}}^{H}\right]\underline{X}$$

II) in-frequency

DFT

$$\underline{Y} = \underline{\underline{B}}\,\underline{X}$$

$\mathrm{diag}(W_N^{-km})$

מטריצה אלכסונית

H.W.
Prove that $\underline{\underline{B}}$ is the same for two cases

PARSEVAL THEOREM

April 27, 1755 – August 16, 1836

- It originates from a 1799 theorem about series by Marc-Antoine Parseval, which was later applied to the Fourier series. It is also known as **Rayleigh's energy theorem**, or **Rayleigh's identity**, after John William Strutt, Lord Rayleigh.

$$\boxed{\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]}$$

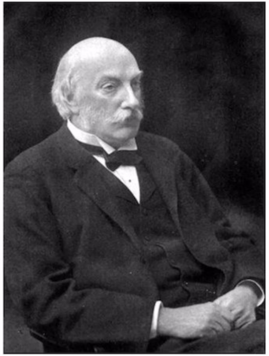
Special case: $$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

PARSEVAL

- Marc-Antoine Parseval des Chênes (27 April 1755 – 16 August 1836) was a French mathematician, most famous for what is now known as Parseval's theorem, which presaged the unitarity of the Fourier transform.

- **Scientific career**: mathematics

LORD RAILEIGH



- John William Strutt, 3rd Baron Rayleigh, (12 November 1842 – 30 June 1919) was an English mathematician who made extensive contributions to science. He spent all of his academic career at the University of Cambridge.
- 1904 Laureate of the **Nobel Prize in Physics** "for his investigations of the densities of the most important gases and for his discovery of argon in connection with these studies."
- **Scientific career**: physics, optics, acoustics
- Rayleigh provided the first theoretical treatment of the **elastic scattering of light by particles much smaller than the light's wavelength**, a phenomenon now known as "Rayleigh scattering", which notably explains why the sky is blue.