DISCRETE FOURIER TRANSFORM (DFT): CIRCULAR CONVOLUTION, ZERO-PADDING

Prof. Alina Karabchevsky, www.alinakarabchevsky.com

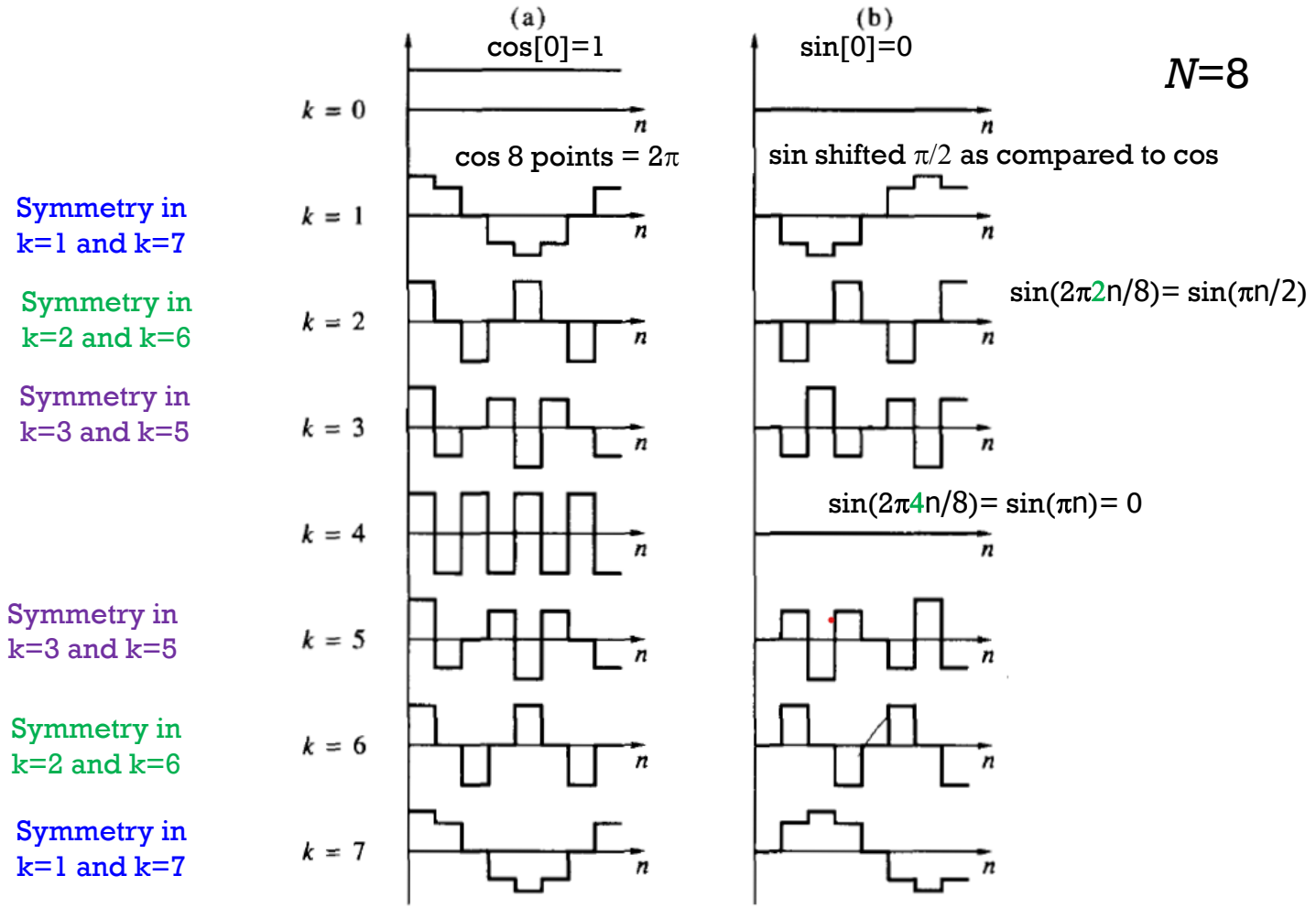
Introduction to Signal Processing,

School of ECE,

Ben-Gurion University

Reading: Chapter 4 by B. Porat; Chapter 8 by Oppenheim and Shafer.

DEMONSTRATION OF DFT



8 points = 2π
2 points = π/2

4 points = 2π
1 points = π/2

Note: via this basis of cosines and sines, one can represent any vector of $\mathbb{C}^N$

Figure 4.5 The DFT basis vectors for $N = 8$: (a) real part; (b) imaginary part.

PARSEVAL THEOREM

$$\boxed{\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]}$$

Plancherel theorem: $$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

the sum (or integral) of the square of a function is equal to
the sum (or integral) of the square of its transform

PARSEVAL THEOREM: PROOF WITH MATRIX FORMULATION

$$\boxed{\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]}$$

Matrix formulation: $\underline{y}^H \underline{x} = \frac{1}{N}\underline{Y}^H \underline{X}$

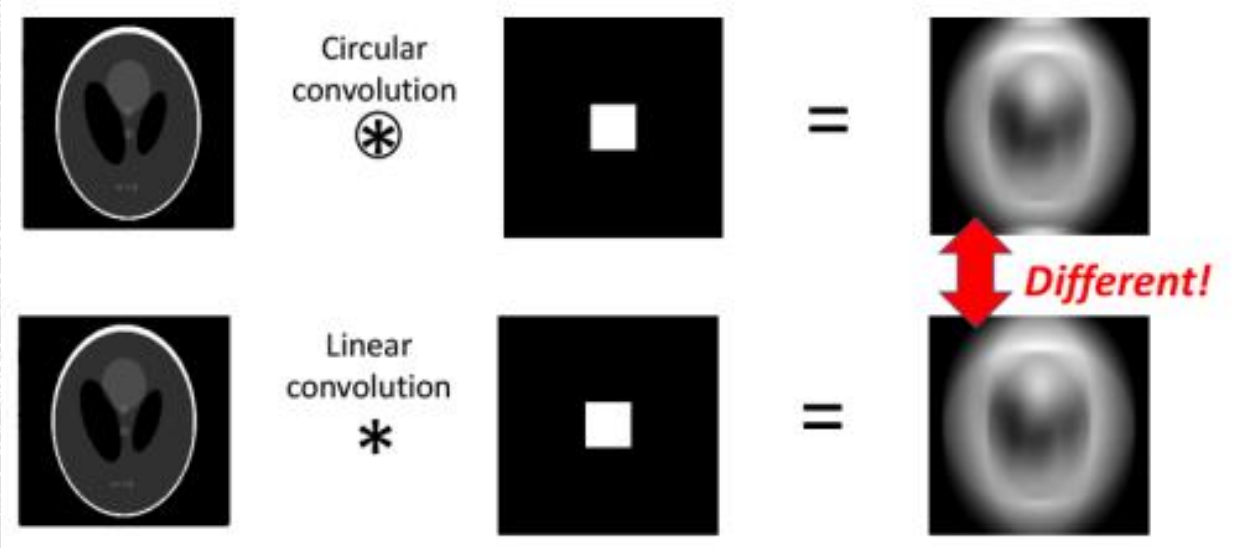
Proof:

$$\frac{1}{N}\underline{Y}^H \underline{X} = \frac{1}{N}\left(\overbrace{\underline{\underline{F}}\,\underline{y}}^{\text{DFT}}\right)^H \underline{X} =$$

$$= \underline{y}^H \overbrace{\frac{1}{N}\underline{\underline{F}}^H \underline{X}}^{\text{IDFT:}} = \underline{y}^H \underline{x}$$

- Note: <u>more simple</u> and computationally efficient as compared to sums.

CIRCULAR CONVOLUTION
קונבולוציה מעגלית



- Note that the linear convolution and circular convolution produce different results (as can be observed near the top and bottom of the images).

CIRCULAR CONVOLUTION

- Assume two signals $\boldsymbol{x_1[n]}$ and $\boldsymbol{x_2[n]}$ and their DFT transforms $\boldsymbol{X_1[k]}$ and $\boldsymbol{X_2[k]}$,

$$n = 0, \dots, N-1; k = 0, \dots, N-1.$$

- Let define the **multiplication in frequency** $X_3[k] = X_1[k] \cdot X_2[k]; k = 0, \dots, N-1$
- $X_3[k]$ has length of N. What is $x_3[n]$?

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k] \cdot X_2[k] e^{j\frac{2\pi}{N}kn}$$ IDFT

$$= \frac{1}{N}\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] e^{-j\frac{2\pi}{N}km} X_2[k] e^{j\frac{2\pi}{N}kn}$$ DFT$\{x_1\}$

$$= \sum_{m=0}^{N-1} x_1[m] \frac{1}{N}\sum_{k=0}^{N-1} X_2[k] e^{-j\frac{2\pi}{N}km} e^{j\frac{2\pi}{N}kn}$$ IDFT of Signal multiplied by exp

$$= \sum_{m=0}^{N-1} x_1[m] x_2\left[\left((n-m)\right)_N\right]$$

this is similar to convolution
but circular one-> due to modulo N

EXAMPLE

- Let assume two sequences of length N=4: $x_1[n] = \{2,1,2,1\}$; $x_2[n] = \{1,2,3,4\}$; $n = 0, \dots, 3$
- Calculate the circular convolution. Note: for each n, the sum below will change
- Solution: $x_3[n] = \sum_{m=0}^{3} x_1[m] x_2\left[\left((n-m)\right)_4\right] \quad n = 0, \dots, 3$

$$x_1[m] = \{2,1,2,1\}$$

We will circulate convolution for n=0:

$$x_2\left[\left((0-m)\right)_4\right] = \{\underset{m=0}{x_2[0]}, \underset{m=1}{x_2[((-1))_4]}, \underset{m=2}{x_2[((-2))_4]}, \underset{m=3}{x_2[((-3))_4]}\}$$

$$= \{x_2[0], x_2[3], x_2[2], x_2[1]\}$$

$$= \{1,4,3,2\}$$

$$x_3[0] = \sum_{m=0}^{3} x_1[m] x_2\left[\left((n-m)\right)_4\right] = \{2,1,2,1\}.*\{1,4,3,2\} = 2*1+1*4+2*3+1*2 = 14$$

Solve for n=1,2,3 $\quad to\ get \rightarrow \quad x_3[n] = \{14,16,14,16\}$

CIRCULAR CONVOLUTION

$$z[n] = \{x \circledast y\}[n] = \sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N], \quad 0 \le n \le N-1. \tag{4.51}$$

Other names for this operation are *cyclic convolution* and *periodic convolution.*

The circular convolution of two length-N sequences is itself a length-N sequence. It is convenient to think of a circular convolution as though the sequences are defined on points on a circle, rather than on a line. Take, for example, $N = 12$, and imagine the n coordinate as the hour on a watch, with 0 instead of 12. To perform the circular convolution, proceed as follows:

1. Spread the $x[n]$ clockwise, starting at the zero hour.
2. Spread the $y[n]$ counterclockwise, starting at the zero hour (i.e, the point $y[1]$ goes on the 11th hour, and so on).
3. To compute $z[n]$, rotate the sequence of $y[n]$s clockwise by n steps, then perform element-by-element multiplication of the two sequences, and sum.
4. Repeat for all n from 0 through 11.

Figure 4.9 illustrates this procedure using a 6-digit watch. In this figure we use $x[n] = y[n] = n$, so the numbers represent both indices and values of the two sequences. The value of n and the corresponding value of $z[n]$ is shown beneath each position of the watch.

CIRCULAR CONVOLUTION

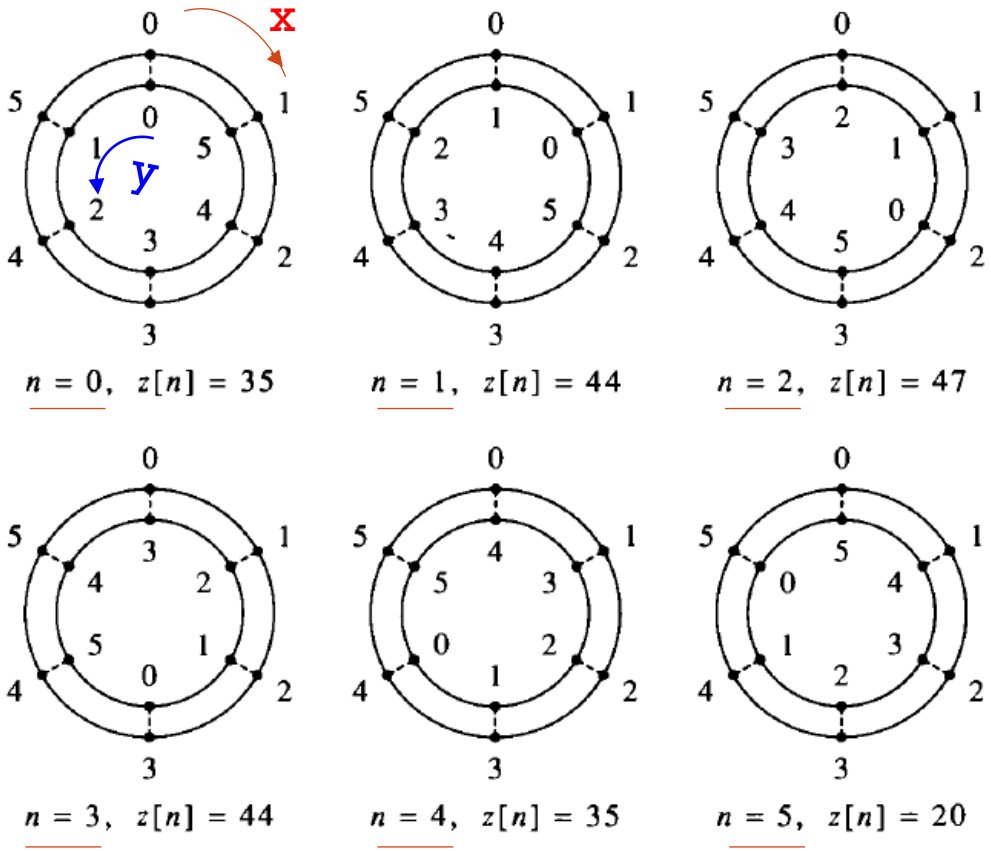


Figure 4.9 Illustration of circular convolution for $N = 6$. Outer circles: $x[m]$; inner circles: $y[n-m]$. The result of the convolution is the sum of products of sequence values near the ends of each dashed line.

CIRCULAR CONVOLUTION FORMULATION WITHOUT MODULO

- We can write the **circular convolution** without modulo, as **linear convolution** of periodic signals:

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

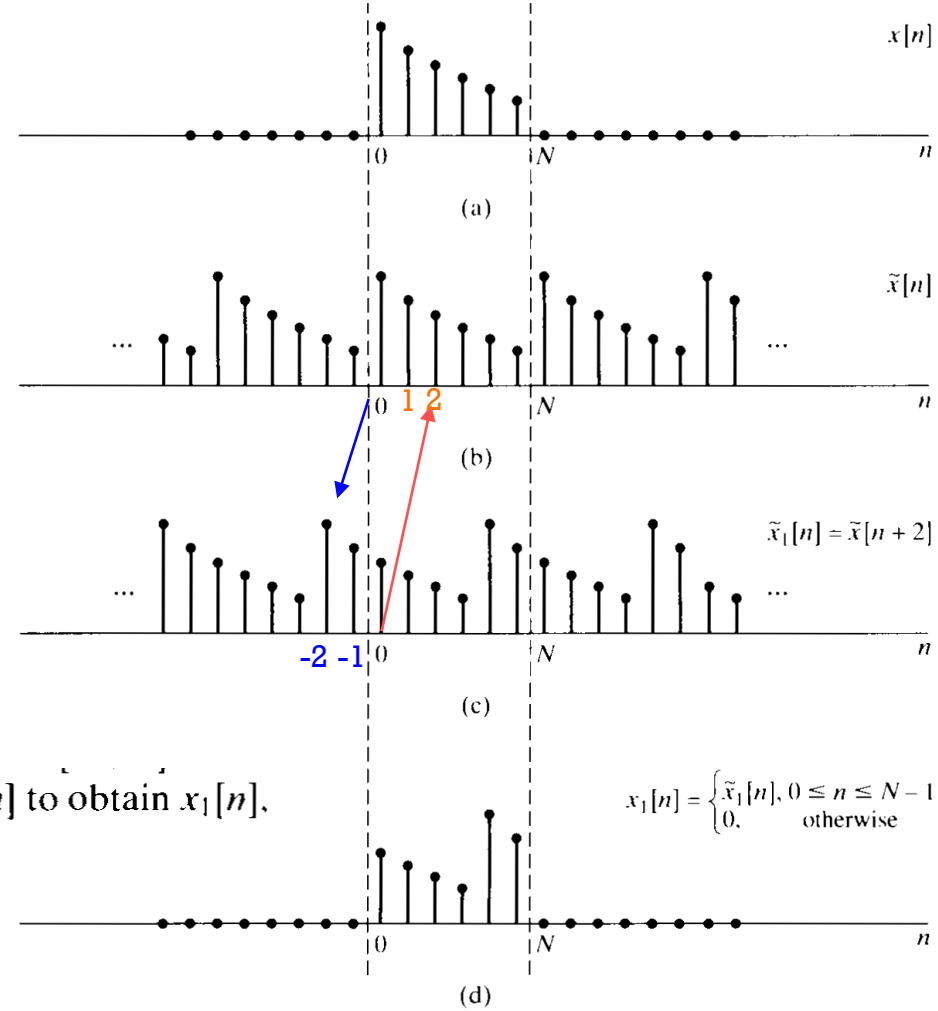
CIRCULAR SHIFT REPRESENTATION AS LINEAR CONVOLUTION ON PERIODIC SIGNALS

$$x_1[n] \doteq x[((n+2))_N] \text{ for } N = 6.$$

m=-2

circular shift

We construct periodic sequence from $x[n]$->>



Shift to the left

we extract one period of $\tilde{x}_1[n]$ to obtain $x_1[n]$.

Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

MATRIX REPRESENTATION OF CIRCULAR CONVOLUTION

- Assume $x[n]$ and $y[n]$ length of N. The outcome of the circular convolution is

$$z[n] = x[n] \textcircled{N} y[n]$$

$\textcircled{N}$ Sign of circular Convolution of modulo N

$$z[n] = \sum_{m=0}^{N-1} x[m] y\left[\left((n-m)\right)_N\right]$$

- Each value of $z[n]$ is the multiplication of $x[n]$ with $y\left[\left((n-m)\right)_N\right]$ circularly shifted.
- We can represent $\underline{z} = \underline{y}\,\underline{x}$

circulant matrix

$m = N-2 \rightarrow 0-(N-2) = 2$

$m = 1 \rightarrow 0-1 = N-1$

Circulant matrix

$$\begin{array}{r} n=0 \rightarrow \\ n=1 \rightarrow \\ \\ \\ n=N-1 \rightarrow \end{array} \begin{bmatrix} z[0] \\ z[1] \\ \vdots \\ z[N-2] \\ z[N-1] \end{bmatrix} = \begin{bmatrix} y[0] & y[N-1] & \dots & y[2] & y[1] \\ y[1] & y[0] & \dots & y[3] & y[2] \\ \vdots & \vdots & \dots & \vdots & \vdots \\ y[N-2] & y[N-3] & \dots & y[0] & y[N-1] \\ y[N-1] & y[N-2] & \dots & y[1] & y[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-2] \\ x[N-1] \end{bmatrix}$$

Example: I matrix is circulant matrix

CIRCULANT MATRIX

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

In linear algebra, a **circulant matrix** is a square matrix in which all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector. It is a particular kind of Toeplitz matrix.

In numerical analysis, circulant matrices are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved using a fast Fourier transform.

They can be interpreted analytically as the integral kernel of a convolution operator on the cyclic group C_n and hence frequently appear in formal descriptions of spatially invariant linear operations. This property is also critical in modern software defined radios, which utilize Orthogonal Frequency Division Multiplexing to spread the symbols (bits) using a cyclic prefix. This enables the channel to be represented by a circulant matrix, simplifying channel equalization in the frequency domain.

In cryptography, a circulant matrix is used in the MixColumns step of the Advanced Encryption Standard.

MATRIX REPRESENTATION OF CIRCULAR CONVOLUTION IN FREQUENCY DOMAIN

- **Matrix $\underline{\underline{y}}$ is named circulant matrix.**

Example:

1) Present the matrix relation of in frequency domain and compare it to time domain.

2) Analyze matrix $\underline{\underline{y}}$.

Answers: 1) Assume the relation in time domain: $\underline{z} = \underline{\underline{y}}\,\underline{x}$

Multiply by DFT matrix $\underline{\underline{F}}\,\underline{z} = \underline{\underline{F}}\,\underline{\underline{y}} \cdot \underline{x}$

$$\text{IDFT}\,\underline{x} = \frac{1}{N}\underline{\underline{F}}^H \underline{X} \rightarrow \quad \underline{Z} = \frac{1}{N}\underline{\underline{F}}\,\underline{\underline{y}}\,\underline{\underline{F}}^H \underline{X} \rightarrow \underline{\boldsymbol{Z}} = \underline{\underline{\boldsymbol{A}}}\,\underline{\boldsymbol{X}}$$

We define: $\underline{\underline{A}} = \frac{1}{N}\underline{\underline{F}}\,\underline{\underline{y}}\,\underline{\underline{F}}^H$ what should be matrix A?

MATRIX REPRESENTATION OF CIRCULAR CONVOLUTION

- Answer:

From the definition of the **circular convolution** which is the multiplication in frequency domain, we expect to obtain:

$$\underline{Z} = \begin{bmatrix} Y[0] & & & \\ & Y[1] & & \\ & & \ddots & \\ & & & Y[N-1] \end{bmatrix} \underline{X} = \underline{\underline{A}}\,\underline{X}$$

$$\underline{\underline{A}} = \mathrm{diag}\{\underline{Y}\}$$

$\underline{\underline{A}}$ is the diagonal matrix via $Y[k]$ on the diagonal and therefore $Z[k] = Y[k]X[k]$

Prove: $\underline{\underline{F}}\,\underline{\underline{y}}\,\frac{1}{N}\underline{\underline{F}}^H = \mathrm{diag}[\underline{Y}]$

EXAMPLE CONTINUATION

2) We will analyze matrix $\underline{\underline{y}}$:

We begin with the relation that we've shown: $\underline{\underline{A}} = \frac{1}{N}\underline{\underline{F}}\,\underline{\underline{y}}\,\underline{\underline{F}}^H$

We multiply by $\underline{\underline{F}}^H$ from the left and $\underline{\underline{F}}$ from the right

$$\underline{\underline{F}}^H\underline{\underline{A}}\,\underline{\underline{F}} = \frac{1}{N}\underbrace{\underline{\underline{F}}^H\underline{\underline{F}}}_{\underline{\underline{I}}}\cdot\underline{\underline{y}}\,\underbrace{\underline{\underline{F}}^H\underline{\underline{F}}}_{N\underline{\underline{I}}} = N\cdot\underline{\underline{y}}$$

From here $\underline{\underline{y}} = \frac{1}{N}\underline{\underline{F}}^H\underline{\underline{A}}\,\underline{\underline{F}}$

We substitute $\frac{1}{N}\underline{\underline{F}}^H\underline{\underline{F}} = I \Rightarrow \underbrace{\frac{1}{N}\underline{\underline{F}} = (F^H)^{-1}}$ and $\underline{\underline{y}} = \overset{Q}{(F^H)}\,\overset{\Lambda}{\mathrm{diag}(\underline{Y})}\,\overset{Q^{-1}}{(F^H)^{-1}}$

eigen decomposition

H.W. prove direct relation

And $\underline{\underline{A}} = \mathrm{diag}[\underline{Y}]$

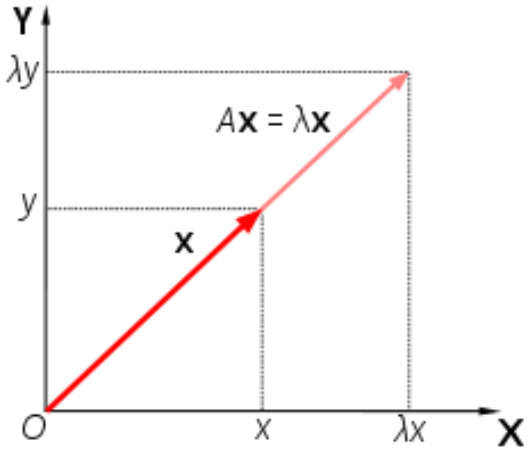
<u>This is eigen decomposition!</u>

$\underline{\underline{y}} = \frac{1}{N}\underline{\underline{F}}^H\underline{\underline{A}}\,\underline{\underline{F}}$

$\underline{\underline{y}} = \underline{\underline{F}}^H\underline{\underline{A}}(\underline{\underline{F}}^H)^{-1}$

מטריצה אלכסונית

EIGEN DECOMPOSITION: REMINDER



Matrix *A* acts by stretching the vector **x**, not changing its direction, so **x** is an eigenvector of *A*.

<u>Reminder:</u>

$$\boxed{\underline{\underline{y}} \cdot \underline{q} = \lambda \cdot \underline{q}}$$ **eigenvalue equation** for the matrix $\underline{\underline{y}}$

$$\underline{\underline{y}}\,\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}$$

$$\underline{\underline{y}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^{-1}$$

diagonal

$\underline{q}$ - eigenvector וקטור עצמי

λ – eigenvalue ערך עצמי

$\underline{\underline{Q}}$ - eigen matrix מטריצה עצמית

- From the previous developments, IDFT matrix $\underline{\underline{F}}^H$ is matrix of eigenvectors of $\underline{\underline{y}}$ while the eigenvalues are the values of DFT vector $Y[k]$.

EIGEN DECOMPOSITION

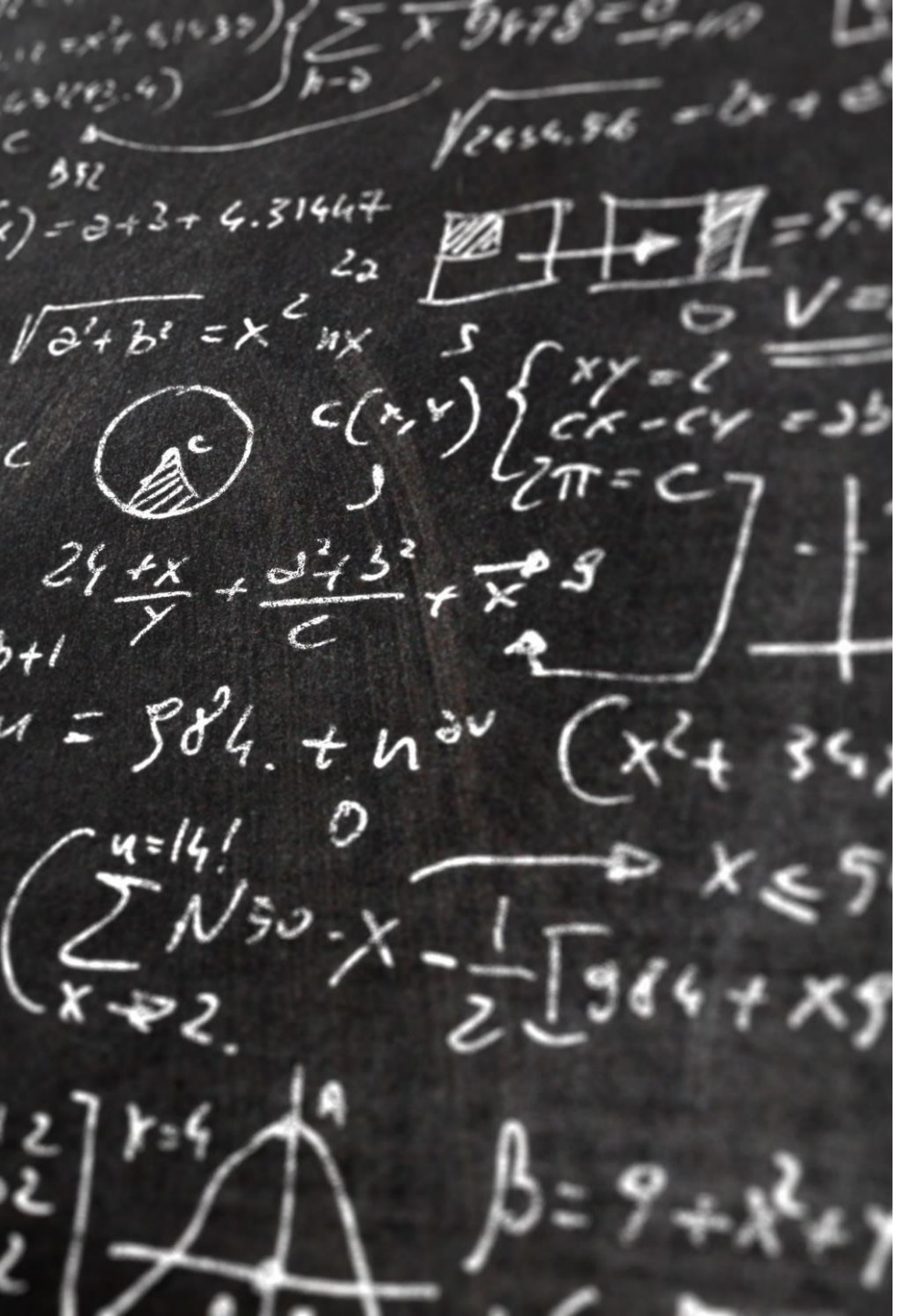
- In general, the decomposition $\underline{\underline{B}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^{-1}$ shows the action of matrix $\underline{\underline{B}}$ on vector $\underline{x}$, meaning $\underline{\underline{B}}\,\underline{x}$ as a first transformation by $\underline{\underline{Q}}^{-1}$ matrix, weight in λ_i and the transformation back by $\underline{\underline{Q}}$

- This representation decomposes the matrix $\underline{\underline{y}}$ to 3 operations:

1) Transformation/projection via $\left(\underline{\underline{F}}^H\right)^{-1} = \underline{\underline{Q}}^{-1}$ do DFT
2) Weighing with Λ -multiply each element by $Y[k], \lambda$ multiply by Y[k]
3) Transformation back $\underline{\underline{Q}} = \underline{\underline{F}}^H$ which is an inverse transform do IDFT

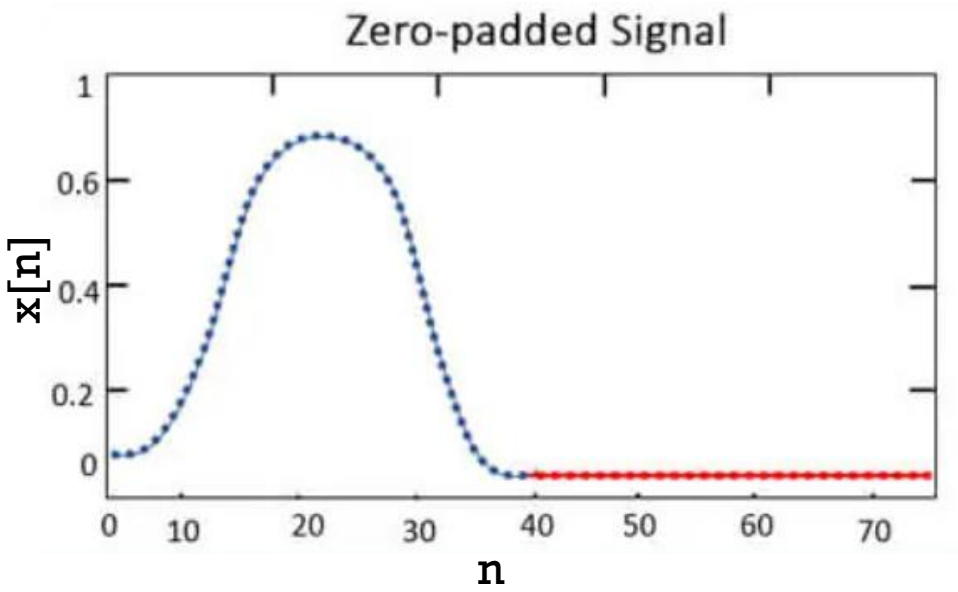
And so, **circular convolution** in time is the multiplication of two DFTs in frequency domain->

-> therefore, eigen decomposition shows the meaning in frequency domain of circular correlation matrix



ZERO PADDING IN-TIME

ריפוד באפסים בזמן



- In in-time zero padding, we **add zeros to the (right) end** of the input sequence.
- Note: adding zeros to the beginning (left) of the input sequence is different operation: it is shift meaning a change in phase

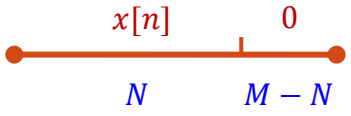
ZERO PADDING

Matlab: `y = [x, zeros(1,M-length(x))];`

Assume $x[n]$ length N. Calculate DFT of length $M > N$

$$x_a[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le M-1. \end{cases} \tag{4.44}$$

from the right!



The operation of adding zeros to the tail of a sequence is called *zero padding*. The DFT of the zero-padded sequence $x_a[n]$ is given by

the tail is zeros

$$X_a^d[k] = \sum_{n=0}^{M-1} x_a[n] \exp\left(-\frac{j2\pi kn}{M}\right) = \sum_{n=0}^{N-1} x[n] \exp\left(-\frac{j2\pi kn}{M}\right) \overset{\text{DTFT}}{=} X(e^{j\theta})\Big|_{\theta=\frac{2\pi k}{M}} \tag{4.45}$$

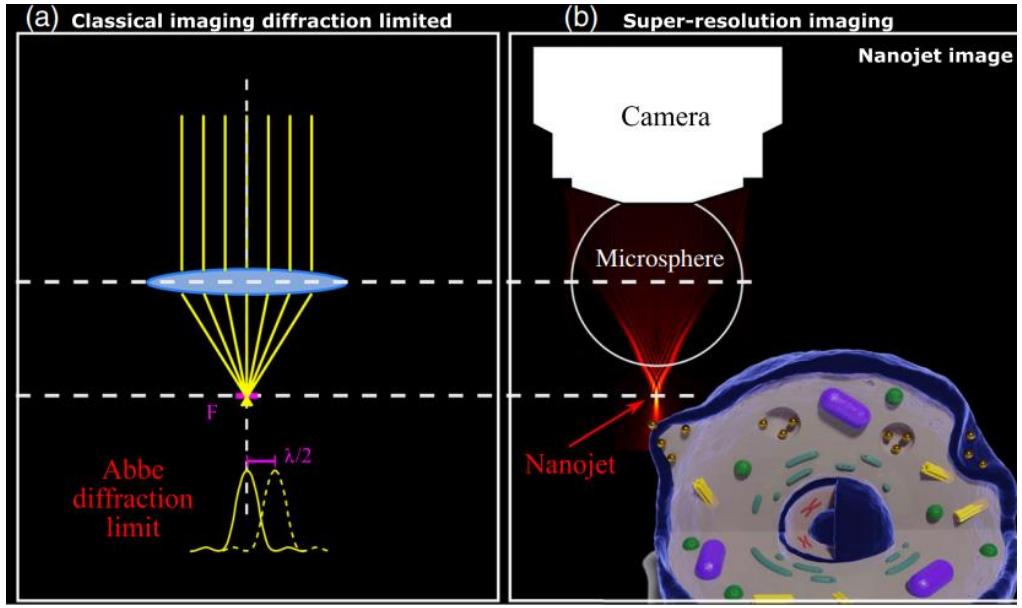
where

$$0 \le k \le M-1. \tag{4.46}$$

Zero padding in time improves the resolution in frequency by making the sampling of DTFT denser but **does not add an information** – simply the result looks better!

WHAT IS RESOLUTION?

- In common language, the word ‘resolution’ may generally be defined as the “**action or process of separating or reducing something into its constituent parts**” (The American Heritage Dictionary of the English Language).
- History: In optics: the power of a microscope system to discriminate the constituent parts of an object down to a certain level of distinction.
- Example 1: The display resolution or display modes of a digital television, computer monitor or display device is the number of distinct pixels in each dimension that can be displayed.
- Example 2:, resolution criteria based on the Nyquist theorem may be very useful to describe the power of a microscope approach to analyze the structure of a completely unknown object.



Super-Resolution Imaging and Optomechanical Manipulation Using Optical Nanojet for Nondestructive Single-Cell Research,

Karabchevsky et Al *Adv. Photonics Res.* **2021**, 2100233

RESOLUTION OF THE FREQUENCY
רזולוציית התדר

- Frequency resolution $\Delta\theta^{DFT} = \frac{2\pi}{N}$
- $\Delta f = \frac{\Delta\theta}{2\pi} = \frac{\Delta\theta^{DFT}}{2\pi T} = \frac{1}{NT}$

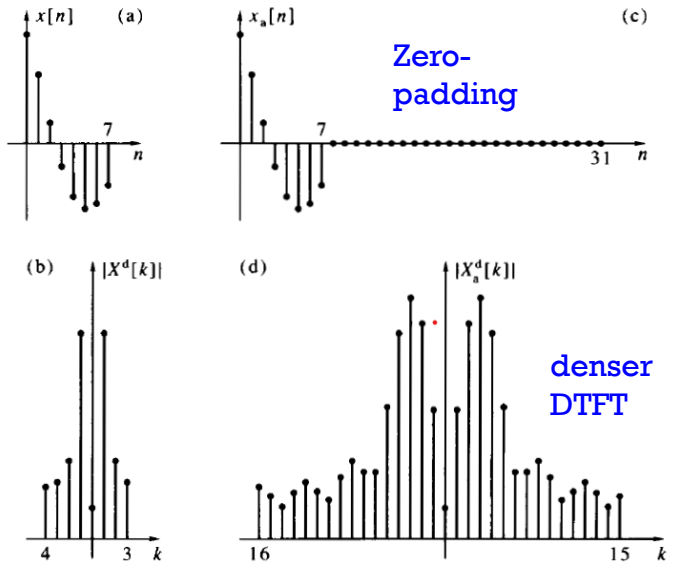
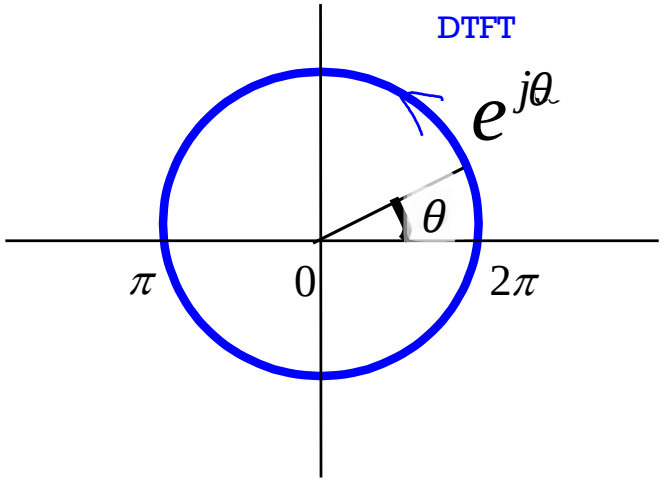
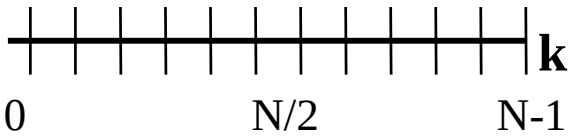


Figure 4.7 Increasing the DFT length by zero padding: (a) a signal of length 8; (b) the 8-point DFT of the signal (magnitude); (c) zero padding the signal to length 32; (d) the 32-point DFT of the zero-padded signal.

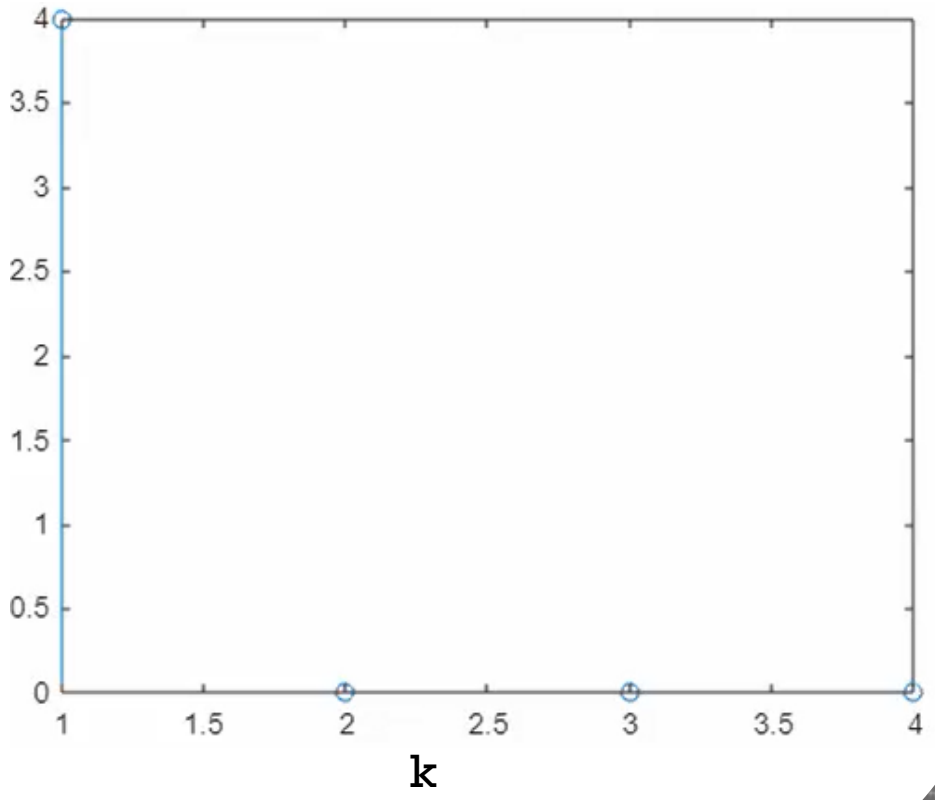
השוואה בין תדר בדיד לתדר זוויתי:





איור 1 – השוואה בין תדר בדיד לזוויתי

MATLAB N=4

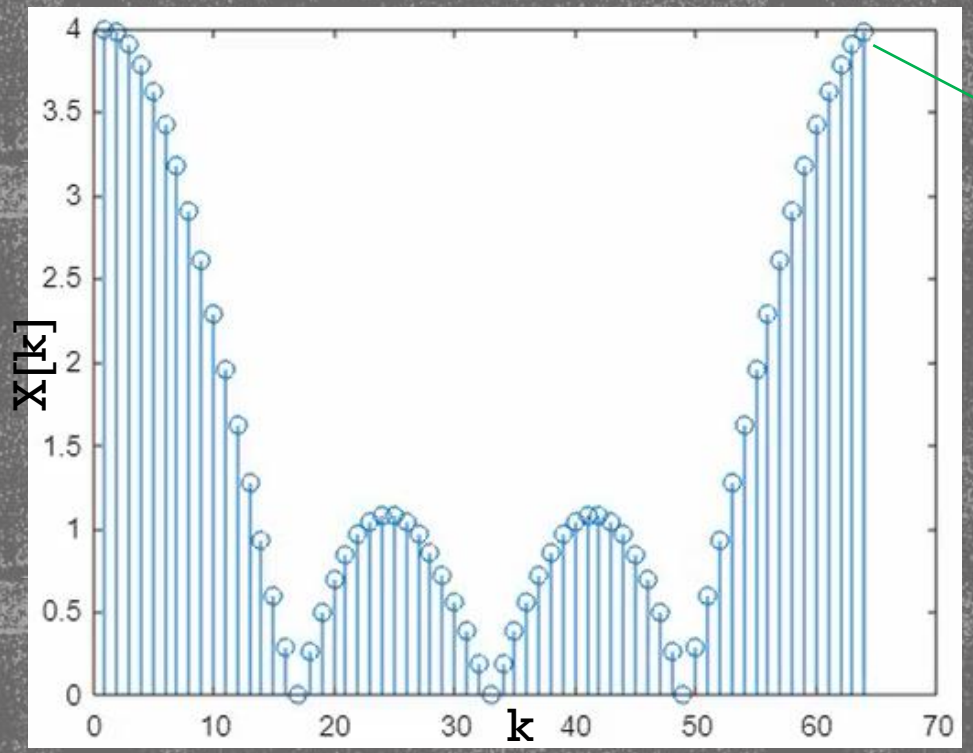



```
>> x=[1,1,1,1];
>> X=fft(x,4);
>> stem(abs(X))
```

add zeros till 64

```
>> X=fft(x,64);
>> stem(abs(X))
```

ZERO PADDING: MATLAB N=64



due to the periodicity+ smoothness treatment

SURVEY: ZERO PADDING

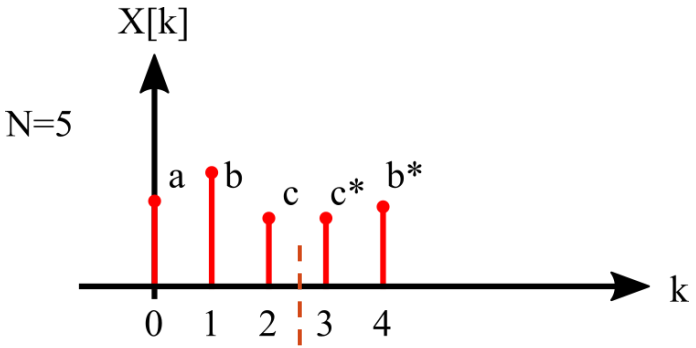
- EasyPolls:

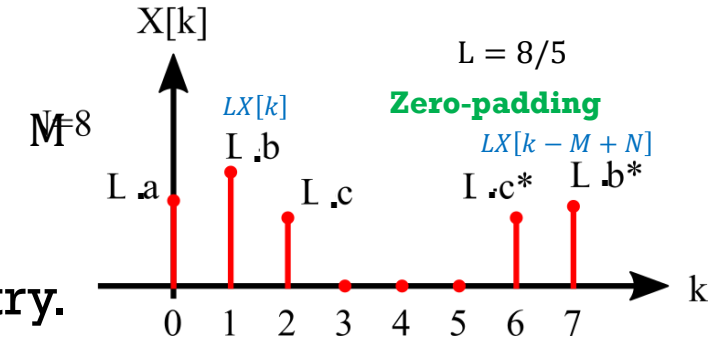
x is of length N. y=[x zero-padded to length 2N]. Which Y=DFT{y} is true for k=0,...,N-1?

- ○ Y[2k]=0
- ○ Y[k]=X[2k]
- ○ Y[2k]=X[k]
- ○ Y[k]=X[k/2]

results vote

ZERO PADDING IN FREQUENCY





In case of the real sequence x$[n]$ in-time, then there is a symmetry in frequency:

$$X[k] = X^*[N-k]$$

Zero-padding in the center of the transform can preserve this symmetry.

Example: Assume we are given x[n] **length N odd**, with DFT $X[k]$. Zero-pad the frequency till the length $\boldsymbol{M = LN}$

$$X_i[k] = \begin{cases} LX[k], & 0 \le k \le \dfrac{N-1}{2} \\ LX[k-M+N], & M - \dfrac{N-1}{2} \le k \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

ZERO PADDING IN FREQUENCY DOMAIN

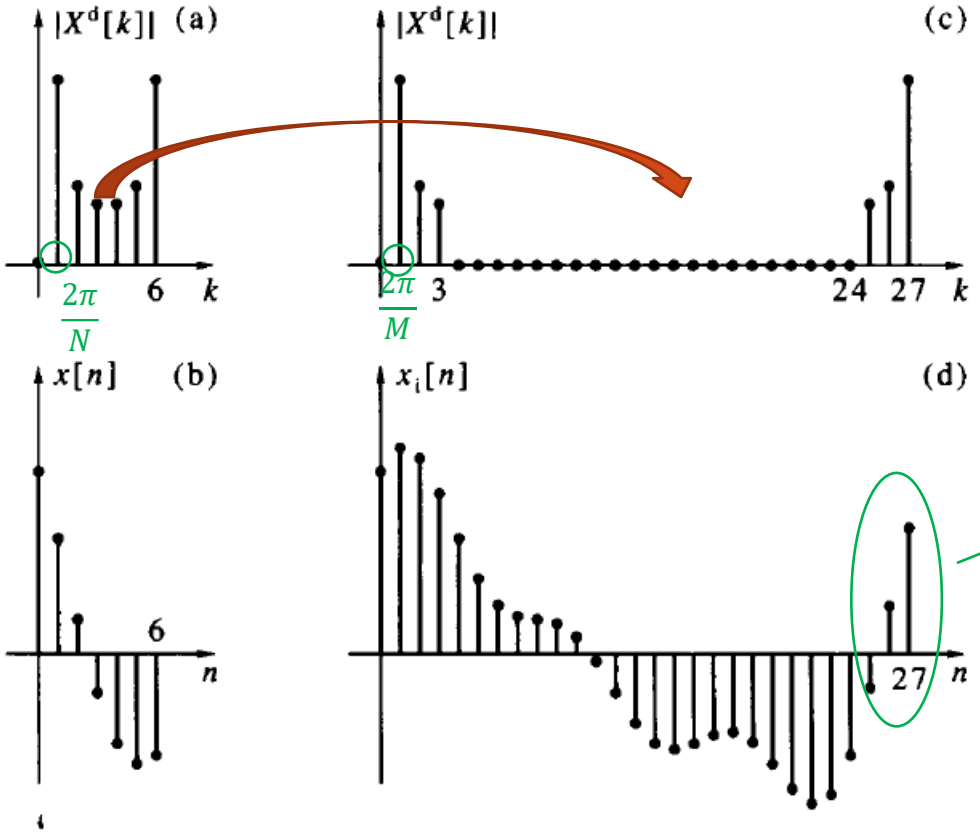
DFT
IDFT
periodic sequences

$$\Delta\theta^{DFT} = \frac{2\pi}{N}$$

2π

distance between two points in frequency $\Delta f = \frac{f_s}{N} = \frac{f_{s_i}}{M}$

new sampling frequency $f_{s_i} = f_s \frac{M}{N}$



due to the periodicity+ smoothness treatment

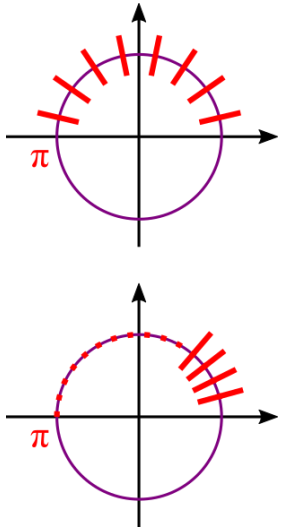


Figure 4.8 Interpolation by zero padding in the frequency domain: (a) DFT of a signal of length 7; (b) the time-domain signal; (c) zero padding the DFT to length 28; (d) the time-domain signal of the zero-padded DFT.

H.W.: check what happens in case N is even.

ZERO PADDING IN FREQUENCY DOMAIN

- To develop an expression for the signal in time:

$$x_i[m] = \frac{1}{M}\sum_{k=0}^{M-1} X_i[k]e^{j\frac{2\pi km}{M}}$$

Solution: H.W.

"sinc"

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\frac{\sin(\pi[m-nL]/L)}{\sin(\pi[m-nL]/M)}$$

This reminds the **interpolation of Shannon**, therefore leads to the increase of resolution in time.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

ZERO PADDING: MATRIX FORM

- Let assume $x[n]$ vector of length N.
- This vector is zero-padded till the length M to obtain $x_z[n]$. We will find the relation between $\underline{x}$ to $\underline{x_z}$:

$$\underset{M\times 1}{\underline{x_z}[n]} = \begin{bmatrix} \underset{N\times N}{\underline{\underline{I}}} \\ \underset{(M-N)\times N}{\underline{\underline{0}}} \end{bmatrix} \underset{N\times 1}{\underline{x}}$$

- In general $\underline{x} \in \mathbb{C}^N$, $\underline{x_z} \in \mathbb{C}^M$, but as compared to $\underline{x_z}$ is included only in sub-space of $\mathbb{C}^M$ since its last $M-N$ elements are zeros.

DFT matrix

- What is the matrix relation between $\underline{x}$ to $\underline{X_z}$: $\underline{X_z} = \underset{M\times M}{\underline{\underline{F}}}\,\underline{x_z} = \underline{\underline{F}} \underbrace{\begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{0}} \end{bmatrix}}_{M\times N} \underline{x} = \underset{M\times N}{\underline{\underline{\tilde{F}}}}\,\underline{x}$

- Here $\underline{\underline{\tilde{F}}}$ is the Fourier matrix size of $M{\times}N$ but not the square matrix which includes only N left columns out of $\underline{\underline{F}}$.
- Is $\underline{X_z}$ in space $\mathbb{C}^M$ or in its sub-space? Answer: in sub-space, since there are only N vectors but nor M vectors to span sub-space.

EXERCISE: ZERO PADDING IN FREQUENCY

- Let assume real vector $\underline{x}$ of length N. Calculate matrix $\underline{\underline{A}}$: $\underline{x_i} = \underline{\underline{A}}\,\underline{x}$ in a way that $\underline{x_i}$ will include elements of $x[n]$ (vector $\underline{x}$) after the **circular interpolation** and new samples are added in between the samples. $\underline{x_i}$ length M.
- How to calculate $\underline{\underline{A}}$?
- Answer: zero padding in frequency.

$$\underline{x_i} = \overbrace{\frac{1}{M}\underline{\underline{F_M^H}} \cdot \underline{\underline{Z}} \cdot \underline{\underline{F_N}}}^{\underline{\underline{A}}}\,\underline{x}$$

מטריצת
באורך IDFT
חדש M

מטריצת
ריפוד באפסים
בתדר
MxN

התמרת
באורך DFT
N

EXAMPLE: N=3, M=6, CALCULATE ELEMENTS OF MATRIX A

- Matrix F_m=F_6 is the DFT matrix of length N=6.
- In MATLAB, we will build matrices $\underline{\underline{F_6}}$, $\underline{\underline{Z}}$, $\underline{\underline{F_3}}$
- We will calculate x_i for different x

$n = [0{:}\,2]; \quad m = [0{:}\,5];$

$F_3 = \exp(-j * 2pi * (n_0)' * n/3)$

Z=definition

$F_6 = \exp(-j * 2pi * (m_0)' * m/6)$

$y = \left(\frac{1}{6}\right) * F_6' * Z * F_3 * x$

stem(real(y));

x=[1;1;1]
[1;0;0]
[0;1;0]

$$X_i[k] = \begin{cases} LX[k], & 0 \le k \le \frac{N-1}{2} \\ LX[k-M+N], & M - \frac{N-1}{2} \le k \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

MATLAB: ZERO PADDING IN FREQUENCY

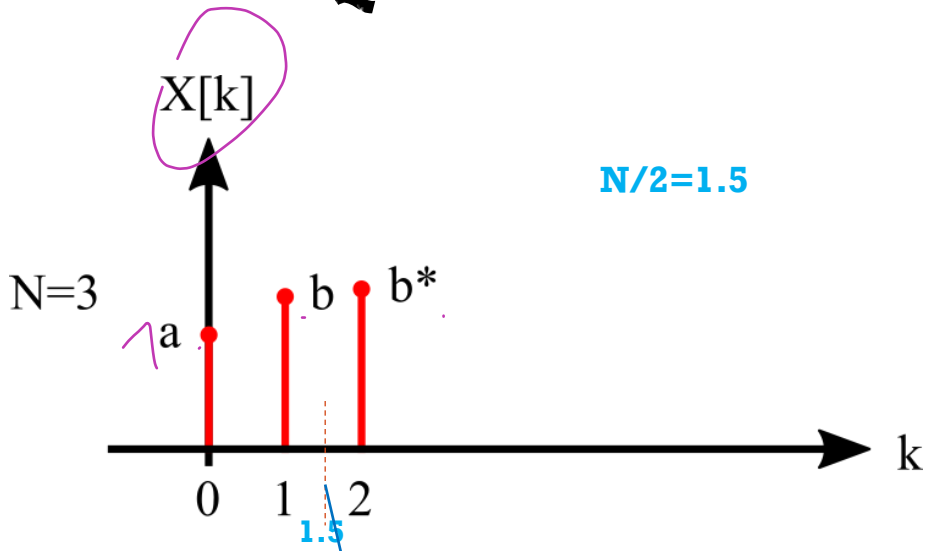
$$\underline{x_i} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{x_i} = \frac{1}{M}\underline{\underline{F_M^H}} \cdot \underline{\underline{Z}} \cdot \underline{\underline{F_N}}\,\underline{x}$$

$$\underline{\underline{F_6}}\,\underline{x_i} = \underline{\underline{Z}} \cdot \underline{\underline{F_3}}\,\underline{x}$$

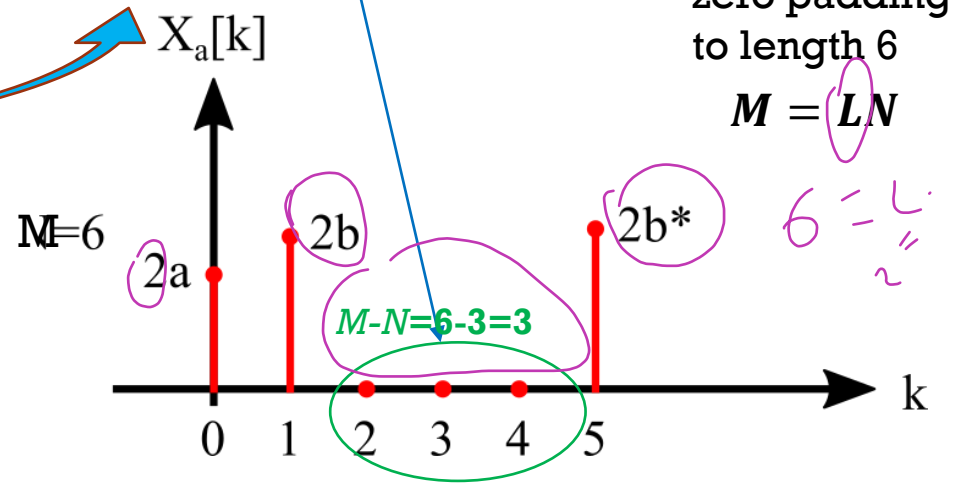
$$\underline{X_a} = \underline{\underline{Z}} \cdot \underline{X}$$

$$\begin{bmatrix} 2a \\ 2b \\ 0 \\ 0 \\ 0 \\ 2b^* \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ b^* \end{bmatrix}$$



zero padding to length 6

$$\boldsymbol{M = LN}$$



EXAMPLE: N=3, M=6, CALCULATE ELEMENTS OF MATRIX A

Zero-padding in frequency

M=6, N=3

Column vector times raw vector

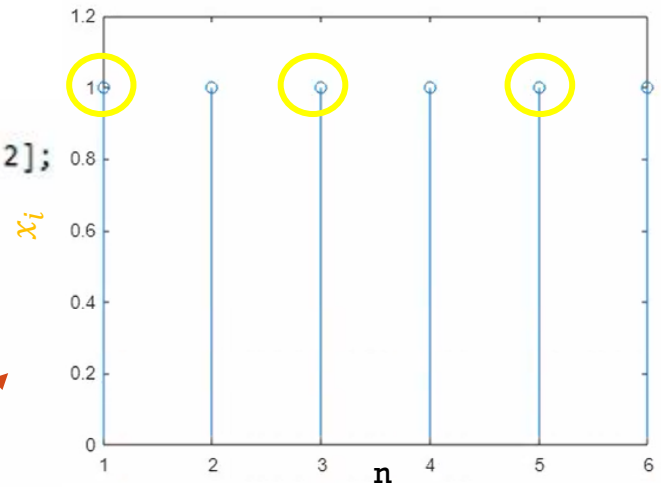
```
>> n=(0:2);
>> m=0:5;
>> F3=exp(-j*2*pi*n'*n/3);
>> Z=[2 0 0;0 2 0 ;0 0 0;0 0 0;0 0 0;0 0 2];
>> Z

Z =

     2     0     0
     0     2     0
     0     0     0
     0     0     0
     0     0     0
     0     0     2
```

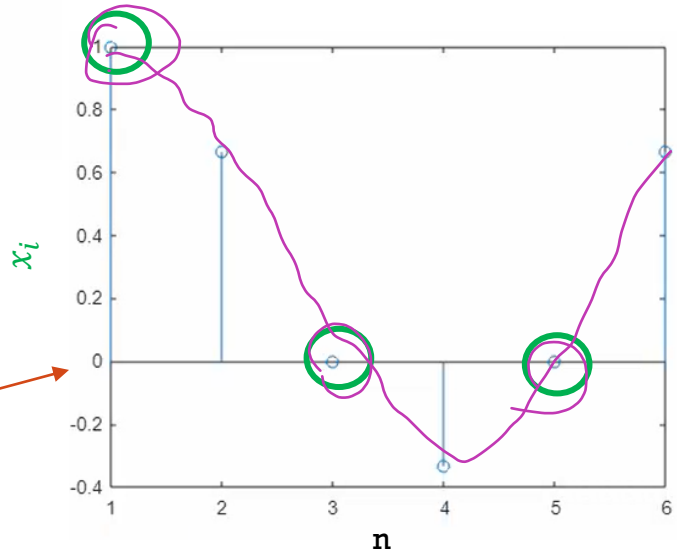
MxN

```
>> F6=exp(-j*2*pi*m'*m/6);
>> A=(1/6)*F6'*Z*F3;
>> x=[1;1;1];
>> xi=A*x;
>> stem(xi)
```

change x-> `>> x=[1;0;0];`

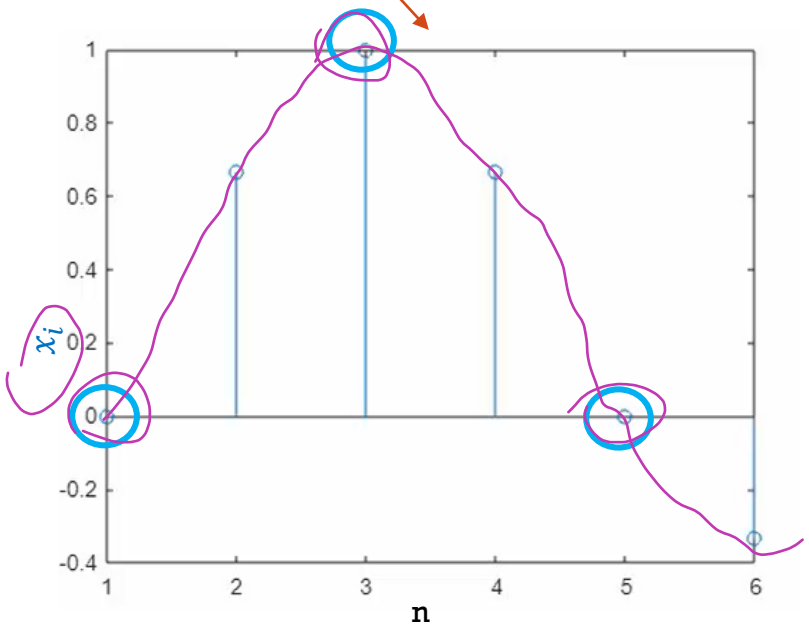
```
>> xi=A*x;
>> stem(xi)
```





change x-> `>> x=[0;1;0];`

```
>> xi=A*x;
>> stem(xi)
```



LINEAR CONVOLUTION BY DFT

- We've seen that **multiplication of two DFTs results in circular convolution** in time.
- In practice, the goal is to calculate the **linear convolution**, for instance the transmission of a signal through the digital system.

$$y[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

Multiplication of DFT results in circular convolution $h\left[\left((n-m)\right)_N\right]$

- Assume that

$$x[n], \qquad n = 0, \dots, L-1$$

$$h[n], \qquad n = 0, \dots, M-1$$

$y[n]$ will be of length $L + M - 1$

From the DTFT properties: linear convolution is converted to multiplication:

$$Y(e^{j\theta}) = X(e^{j\theta})H(e^{j\theta})$$

LINEAR CONVOLUTION BY DFT

- $y[n]$ is of the finite length $N = L + M - 1$. One can represent $y[n]$ as DFT of length $N = L + M - 1$ without loss of information (overlap in time) therefore DFT will be :

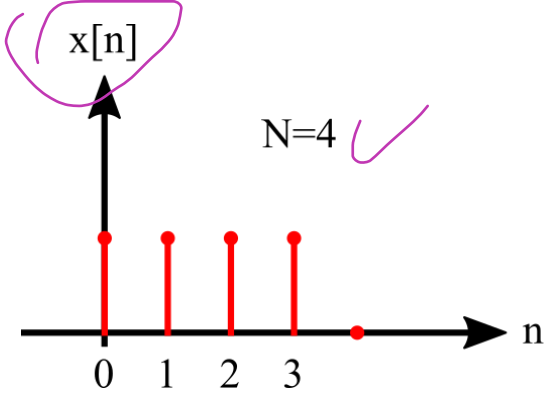
$$Y[k] = Y(e^{j\theta})|_{\theta=\frac{2\pi}{N}k} = X(e^{j\theta}) \cdot H(e^{j\theta})|_{\theta=\frac{2\pi}{N}k} = X[k] \cdot H[k]$$

$N = L + M - 1$

Longer DFT of length $N = L + M - 1$

- Conclusions:
- $X[k], H[k]$ are the denser samples of DTFT are:
  - Sampling of DTFT in $L + M - 1$ samples
  - Zero padding to the length of $L + M - 1$ and calculate the DFT of the zero-padded sequence.
- By multiplying DFT of length $N = L + M - 1$ of x and h and perform the inverse transform to time domain, we will obtain $y[n]$ of length $N = L + M - 1$ but **periodic**. $Y[k]$
- Therefore, **one period=linear convolution.** $Y[k]$ or (y[n]) is the linear conv.

EXAMPLE: LINEAR CONVOLUTION BY DFT

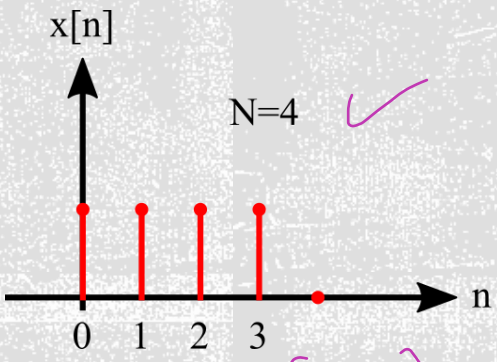


Assuming $\boldsymbol{x[n]} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}\}$

- 1) **linear convolution:** $y[n] = \sum_{m=0}^{L-1=3} x[m]h[n-m]$
- 2) **circular convolution length** $N = 4$: $y[n] = \sum_{m=0}^{3} x_1[m]x_2\left[\left((n-m)\right)_4\right]$
- 3) **circular convolution length** $N = 7$: $y[n] = \sum_{m=0}^{6} x_1[m]x_2\left[\left((n-m)\right)_7\right]$

$N = L+M-1 = 4+4-1$

$$x[n] = \{1, 1, 1, 1\}$$



1) linear convolution:

$$y[n] = \sum_{m=0}^{3} x[m]h[n-m]$$

```
>> x=[1 1 1 1];
>> y=conv(x,x);
>> stem(y)
```

2) circular convolution length $N = 4$:

$$y[n] = \sum_{m=0}^{3} x_1[m]x_2\left[((n-m))_4\right]$$

```
>> X=fft(x);
>> Y=X.*X;
>> y=ifft(Y);
>> stem(y)
```

3) circular convolution length $M = 7$:

$$y[n] = \sum_{m=0}^{6} x_1[m]x_2\left[((n-m))_7\right]$$

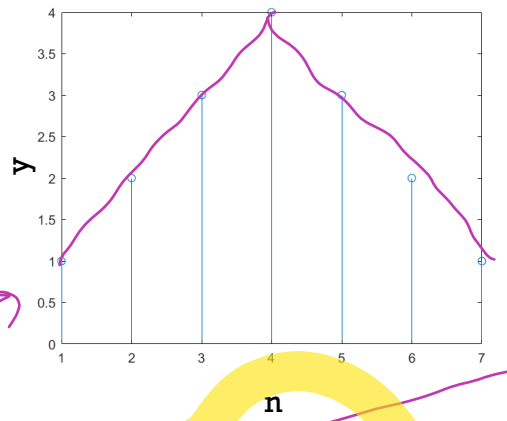
```
>> X=fft(x,7);
>> Y=X.*X;
>> y=ifft(Y);
>> stem(y)
```

4) circular convolution length $M = 27$:

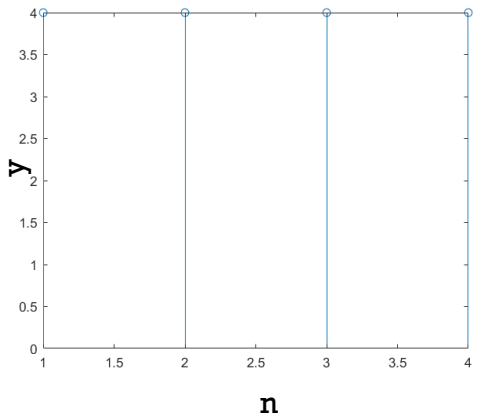
$$y[n] = \sum_{m=0}^{26} x_1[m]x_2\left[((n-m))_{27}\right]$$

```
>> X=fft(x,27);
>> Y=X.*X;
>> y=ifft(Y);
>> stem(y)
```

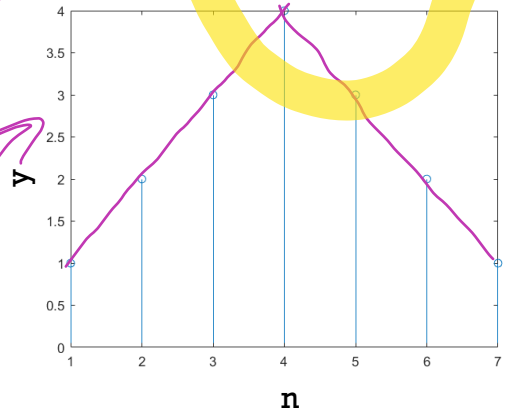
1) Length 4+4-1



2) Length 4

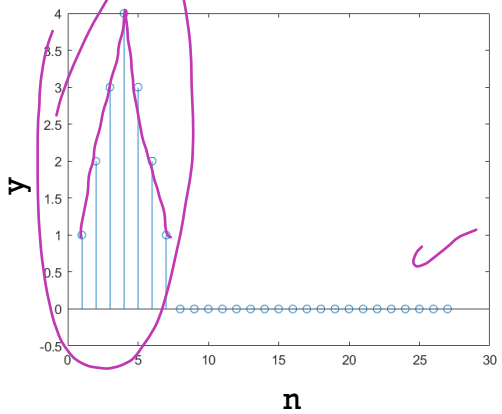


3) Length 7
one period



Linear=cyclic conv

4) Length 27



Linear=cyclic conv

MATLAB: LINEAR CONVOLUTION BY DFT

EXERCISE

- Assuming DTFT pair $x[n] \leftrightarrow X(e^{j\theta})$

  $y[n] \leftrightarrow Y(e^{j\theta})$

  DFT pair $x[n] \leftrightarrow X[k]$

  $y[n] \leftrightarrow Y[k]$

N

y, x

IDFT

DTFT

linear convolution

$$z_1(n) = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\theta})Y(e^{j\theta})e^{j\theta n}\, d\theta$$

circular convolution

IDFT

$$z_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y[k]e^{\frac{2\pi}{N}kn}$$

DFT

1) what are z_1 and z_2 ?

2) When we can calculate z_1 from z_2? Answer: If and only if, $N \geq N_x + N_y - 1$ then DFT will give linear convolution and we can calculate the sequences back in time.

SURVEY: ZERO PADDING

- EasyPolls:

H[k]=[1, 1, 0, 0, 0, 0, 0, 1]. Is the IDFT h[n] the impulse response of an ideal LPF?

- ○ No
- ○ Yes, H[k] is 1 or 0
- ○ Only if zero padded in time to a length of infinity

results **vote**

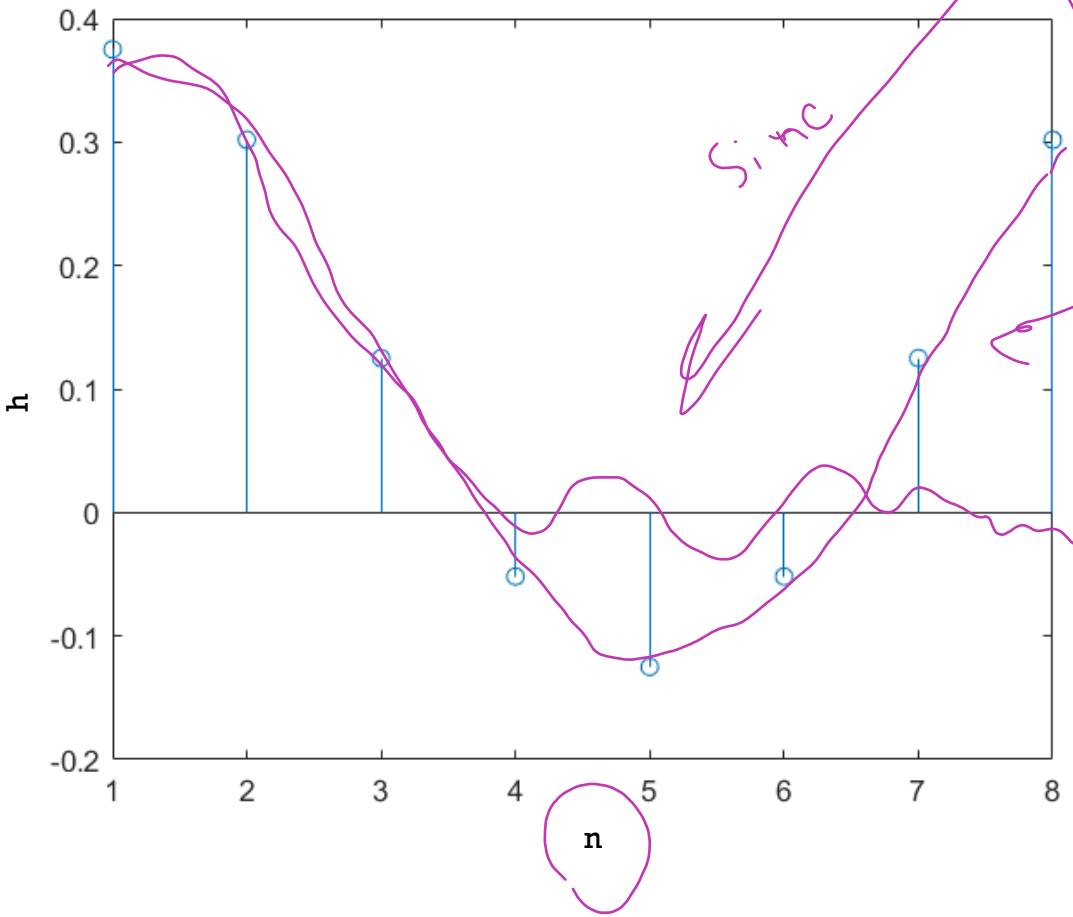
LPF

MATLAB

```
>> H=[1 1 0 0 0 0 0 1];
>> h=real(ifft(H));
>> stem(h)
```

IDFT

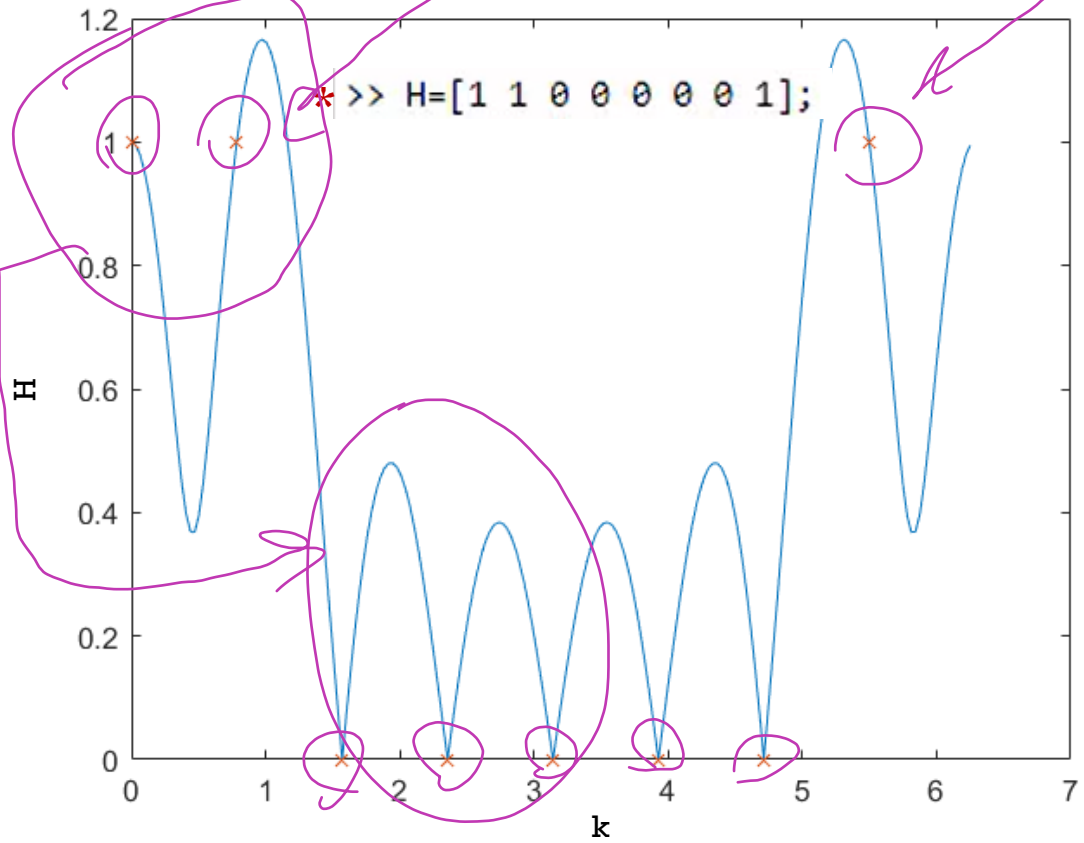
Sinc




```
>> H1=fft(h,200);
>> plot([0:199]*2*pi/200,abs(H1),(0:7)*2*pi/8,abs(H),'x');
```

N=200

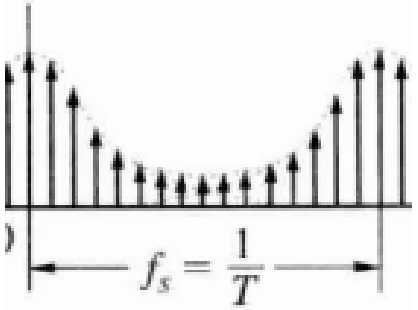
```
>> H=[1 1 0 0 0 0 0 1];
```

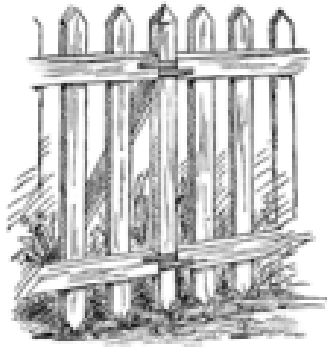


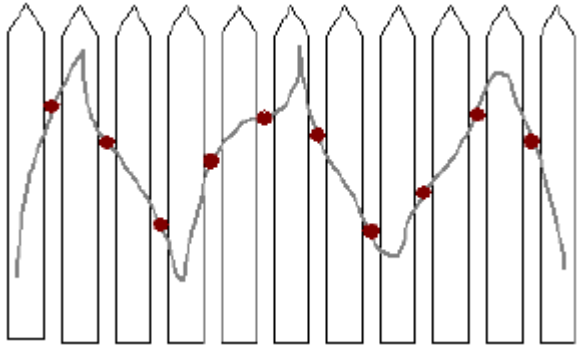
PICKET FENCE EFFECT

- The amount of time sampled, T_0, determines the spacing between frequencies
- We don't know the shape of $X(e^{j\theta})$ before we start
- If we sample the frequencies too far apart, we can miss features in the spectrum
  - This happens with too small a time window
  - T_0 too small

- Numerical computation method yields uniform sampling values of $X(\omega)$.
- Information between samples in spectrum is missing – picket fence effect:
- Can improve spectral resolution by increasing T.







DFT PROPERTIES

TABLE 8.2

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km}X[k]$
6.	$W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{J}m\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.		
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$