LECTURE 2: UNIFORM SAMPLING

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Introduction to Signal Processing Course,

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Reading: Chapter 3 by B. Porat

RELATIONS BETWEEN TRANSFORMS

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s) \qquad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

מתוך I הקשר בין $X_s(j\omega) \leftrightarrow X(j\omega)$

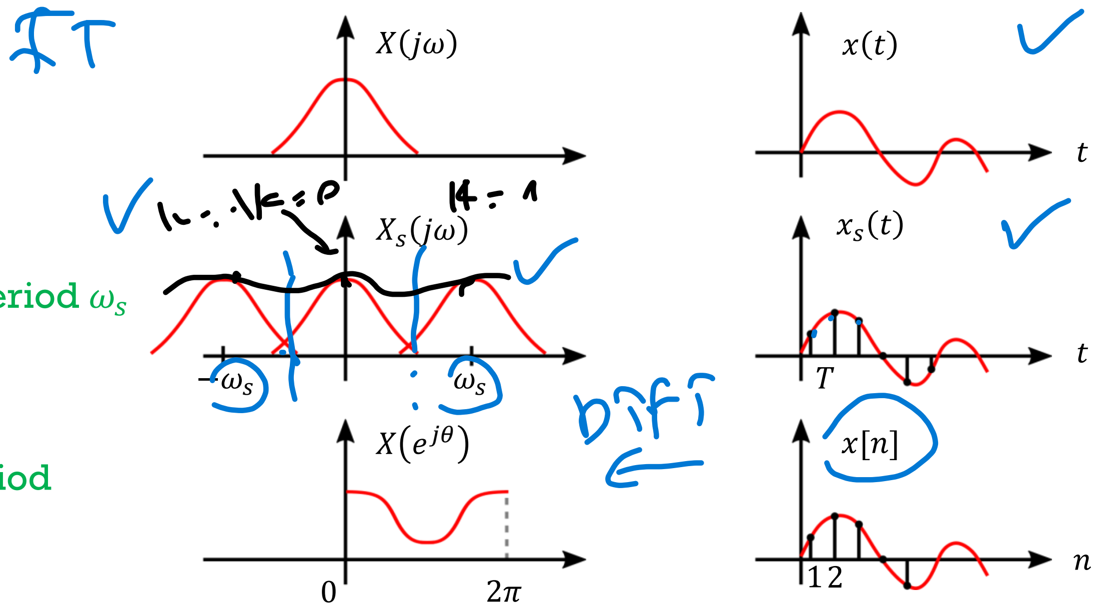
פונ״ק מחזורית

פונ״ק על עיגול= מחזור אחד

$$X_s(j\omega) = X(e^{j\theta})|_{\theta=\omega T}$$

מתוך II הקשר בין $X(e^{j\theta}) \to X(j\omega)$ מיידי.

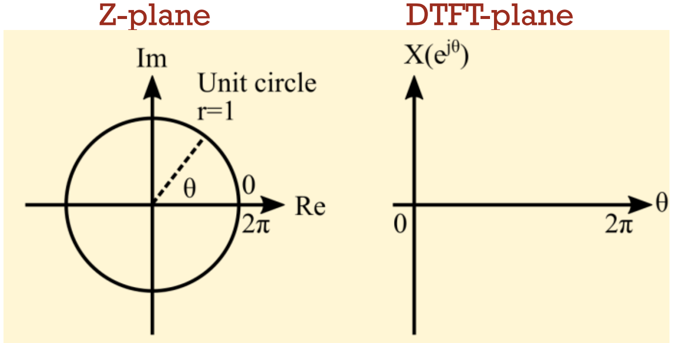
מסקנות: $X_s(j\omega), X(e^{j\theta})$ מורכבים משיכפולים של $X(j\omega)$ כל ω_s



$X_s(j\omega)$ is periodic with period ω_s

$X(e^{j\theta})$ is DTFT = one period

נסכם גרפית:



SAMPLING: CASE I – BANDWIDTH LIMITED SIGNAL SAMPLED AT NYQUIST FREQUENCY

נתון אות **מוגבל סרט**: לתחום $|\omega| \gg \omega_m$ מחוץ לתחום זה $X(j\omega) = 0$. (ראה/י איור א).

נתון כי $\frac{\omega_s}{2} = \frac{\pi}{T}$ (ראה/י איור ב).

$$2\pi f_m = \omega_m \qquad 2\pi f_m \leq \omega_s$$

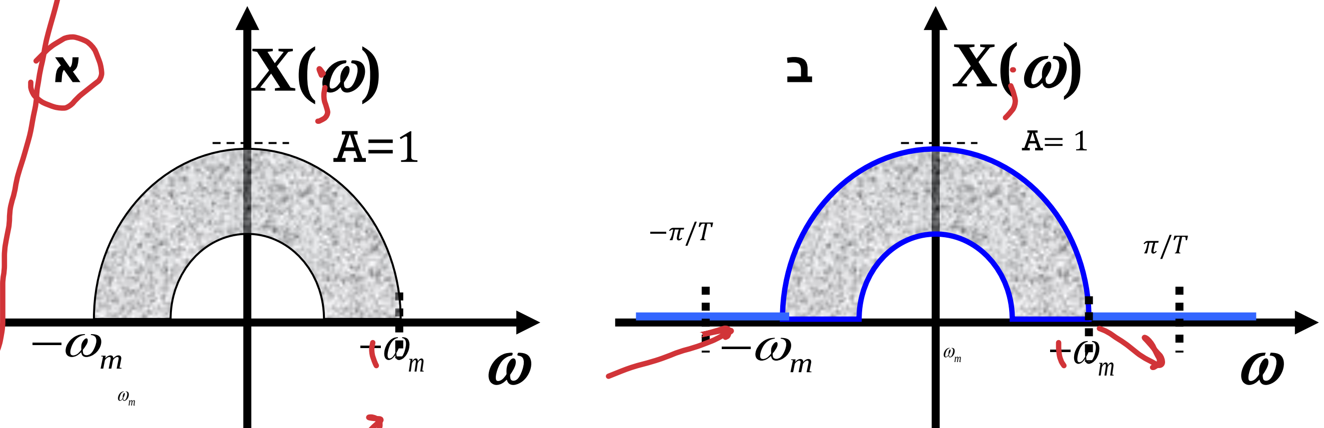
$$\pi f_s = \frac{\omega_s}{2}$$

$$f_s = \frac{1}{T} \rightarrow \frac{\pi}{T} = \frac{\omega_s}{2}$$

We will sample x(t) at frequency $f_s \geq 2f_m$

$f_m \geq \omega_m/2\pi$

בגלל ש $|\omega| < \omega_m < \pi f_s = \frac{\omega_s}{2}$ לא תהיה חפיפה ספקטראלית בין החזרות של $X(j\omega)$



Band-limited signal

SAMPLING: CASE I

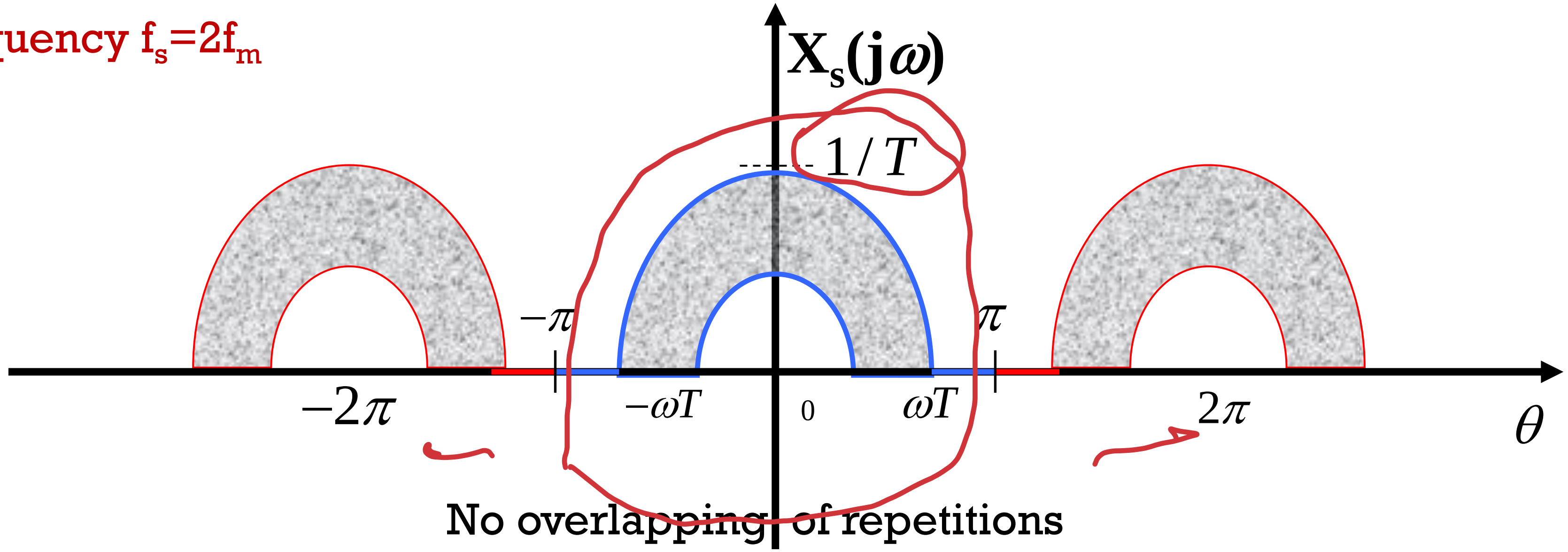
We will sample x(t) at frequency $f_s=2f_m$
$f_m=\omega_m/2\pi$

f_m

נסמן: $\theta = \omega T$, ואז אם תדירות כלשהי היא שווה ל $\omega = \pm\frac{\omega_s}{2}$ נתון כי $\frac{\omega_s}{2} = \frac{\pi}{T}$
כאשר
אז נקבל גבול של θ מוגדר בתחום: $-\pi < \theta < \pi$ כלומר,

$$X\left(e^{j\theta}\right) = \frac{1}{T}X(j\omega)|_{\theta=\omega T},\ -\pi < \theta < \pi$$

Nyquist frequency $f_s=2f_m$



No overlapping of repetitions

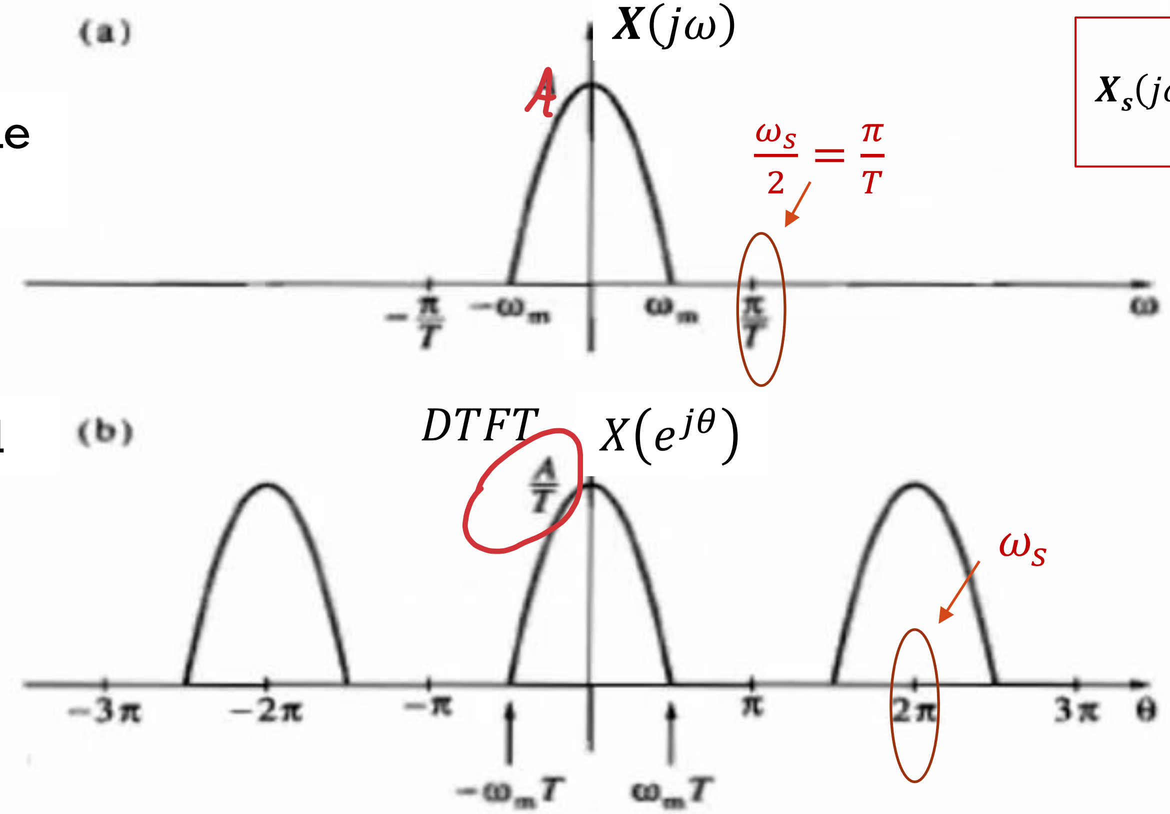
דגימה אידאלית : שיכפול ללא עיוות רק כיווץ והגבה.

SAMPLING OF BAND-LIMITED SIGNAL ABOVE THE NYQUIST RATE: EXAMPLE

$$\boldsymbol{X_s}(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

FT of the continuous-time signal

FT of the sampled signal



Nyquist frequency $f_s=2f_m$

By sampling the bandwidth limited signal at Nyquist frequency or above it, there is no overlapping of repetitions takes place.

$$X(e^{j\theta}) = \frac{1}{T}X(j\omega)|_{\theta=\omega T}, \quad -\pi < \theta < \pi$$

HARRY NYQUIST – HISTORICAL NOTE

Harry Nyquist

- Harry Nyquist established foundations for well-known Nyquist sampling theorem in 1920s.
- There is no loss of information when the analog (continuous-time) signal is sampled at least 2X of its highest frequency.
- The theorem was mainly driven by telegraph communication problems between world war I and II.

[1] H. Nyquist, “Certain Factors Affecting Telegraph Speed,” Journal of the A.I.E
[2] H. Nyquist, “Certain Topics in Telegraph Transmission Theory,” Transactions
Engineers, 1928

Harry Nyquist

Harry Nyquist

Born	February 7, 1889 Nilsby, Stora Kil, Värmland, Sweden
Died	April 4, 1976 (aged 87) Harlingen, Texas, U.S.
Nationality	Swedish
Citizenship	Swedish / American
Alma mater	Yale University University of North Dakota
Known for	Nyquist–Shannon sampling theorem Nyquist rate Johnson–Nyquist noise

EXAMPLE

$$\boldsymbol{X_s}(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

✓ $x(t) = \text{sinc}(f_0 t) \quad \omega_0 = 2\pi f_0$

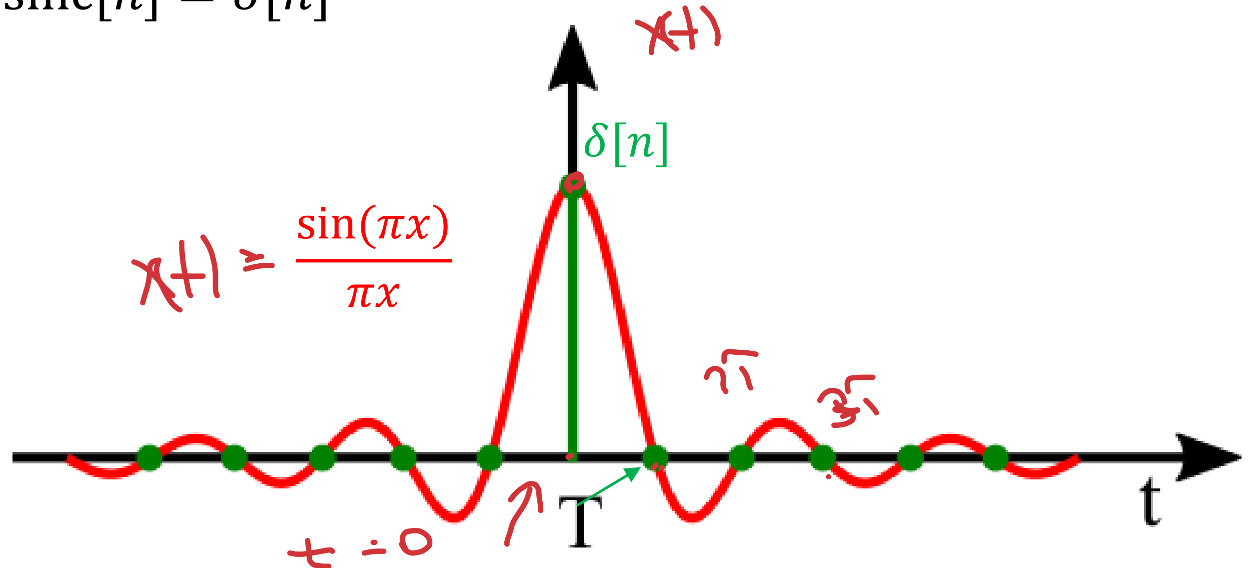
ממשי, זוגי:
סופי

$\underline{X(j\omega)} = T \cdot \Pi(\omega/\omega_0)$ $\Longrightarrow$ from here

$, \omega_m = \frac{\omega_0}{2}$

- We will sample at Nyquist frequency $\omega_s = 2\omega_m = \omega_0, \ T = 1/f_0$
- $x[n] = x(nT) = x\left(\frac{n}{f_0}\right) = \text{sinc}[n] = \delta[n]$

t

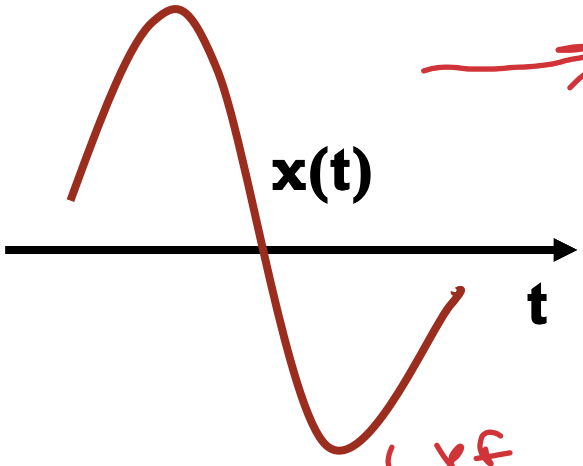


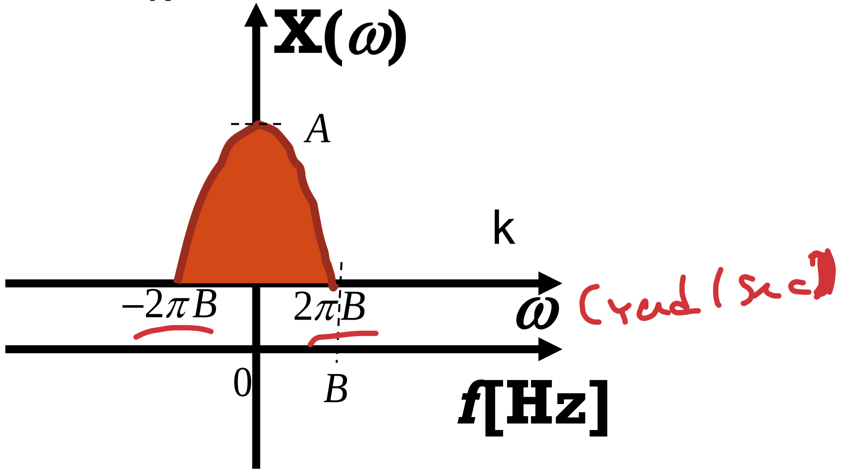
From infinite number of options, only one signal will be bandwidth limited at the right frequency- the sinc signal!

SAMPLING CAUSES *REPETITIONS* OF THE FREQUENCY SPECTRUM

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

- The spacing between repetitions depends on the sampling rate
- We can reconstruct the original signal using a lowpass filter which filters out all the repetitions

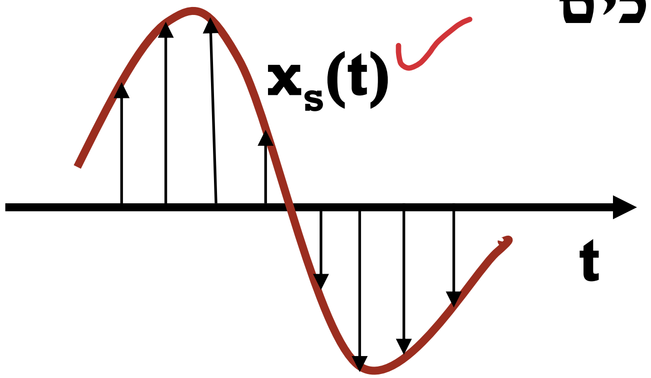


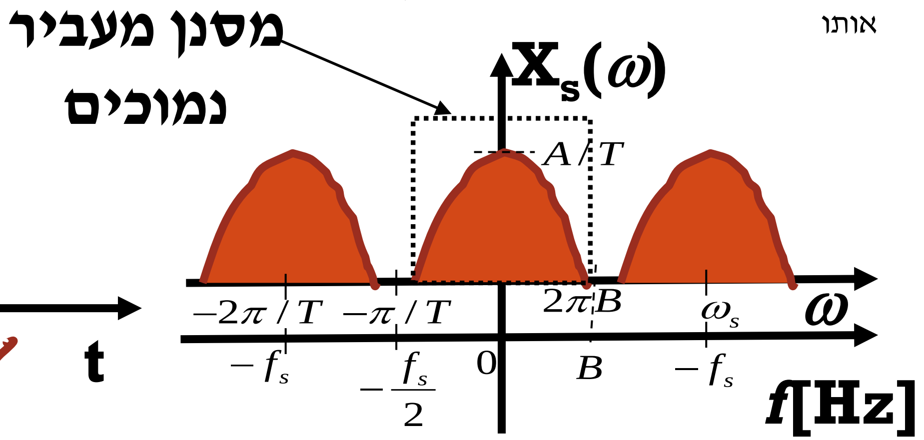


*חסום בזמן אינו חסום בתדר!!

*במציאות אותות תמיד מוגבלי זמן ולכן אינם מוגבלו תדר

*כלומר, תמיד תהיה שגיאת אליאסינג לכן נחפש דרכים למזער אותו

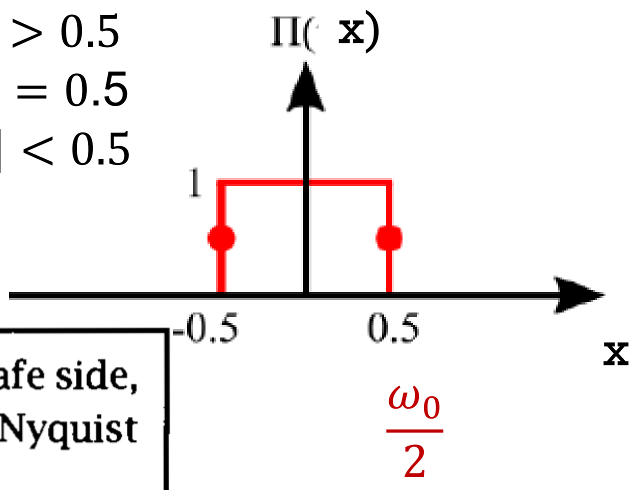




From infinite number of options, only one signal will be bandwidth limited at the right frequency- the sinc signal!

SIGNAL RECONSTRUCTION שחזור

$$\Pi(x) = \begin{cases} 0 & |x| > 0.5 \\ 0.5 & |x| = 0.5 \\ 1 & |x| < 0.5 \end{cases}$$



> - Never sample below the Nyquist rate of the signal. To be on the safe side, use a safety factor (e.g., sample at 10 percent higher than the Nyquist rate).
> - In case of doubt, use an antialiasing filter before the sampler.

- Reconstructing a continuous time signal from samples is called *interpolation*
  - We must **assume** that the original signal was band limited to B Hz < $0.5f_s$
  - We low pass filter with any low pass having a bandwidth between B and f_s-B Hz
    - Usually use $0.5f_s$

$$X(j\omega) = X_s(j\omega) \cdot H(j\omega)$$

פונק' מסנן

The ideal interpolation filter:

$$H(j\omega) = T\,\Pi\left(\frac{\omega}{2\pi f_s}\right) = T\,\Pi\left(\frac{\omega}{\omega_s}\right) \qquad \omega_s = \frac{2\pi}{T}$$

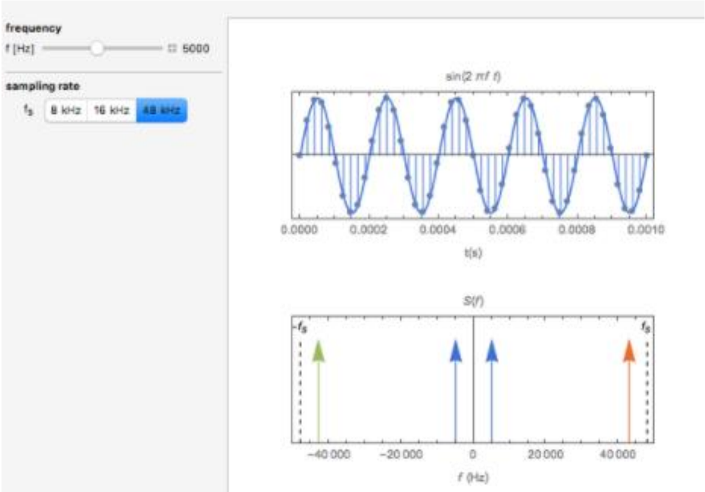
$$h(t) = sinc\left(\frac{t}{T}\right)$$

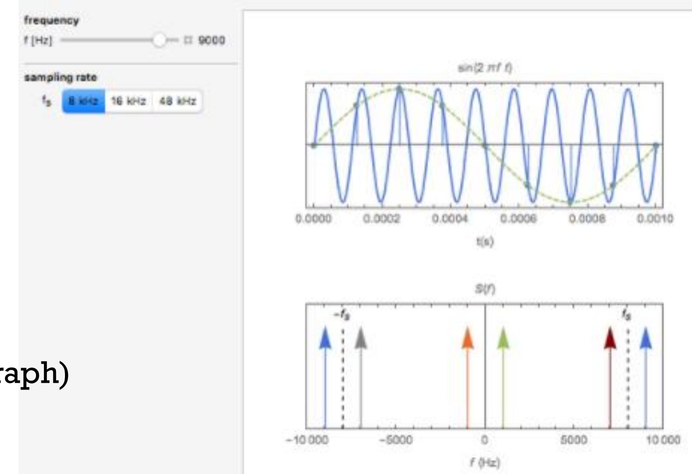
SHANNON-NYQUIST THEOREM (MATHEMATICAL PROBLEM)

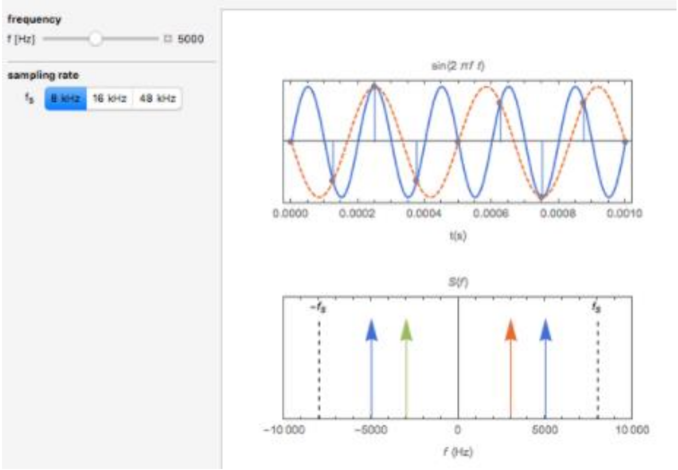
The Shannon-Nyquist theorem states that if a function $x(t)$ contains no frequencies higher than B hertz, then it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

History

formulation date	1928 (93 years ago)
status	proved
proof date	1949 (21 years later) (72 years ago)
prover	Claude Shannon





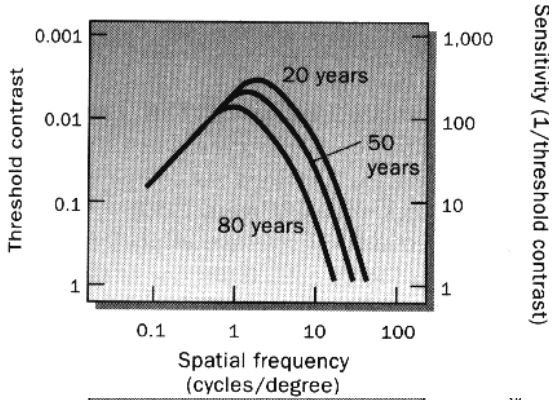


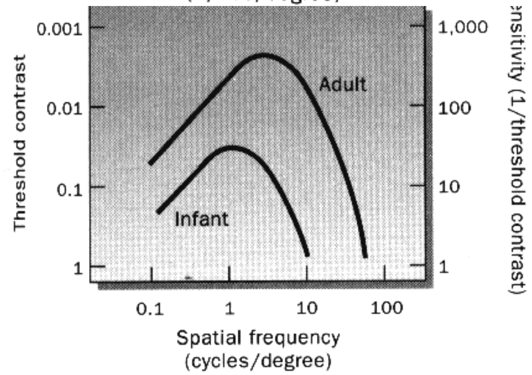
To conclude:
No information loss if:
$2\pi-\omega T_m >= \omega_m T_s$
$\omega_s >= 2\omega_m$
Clode Shannon and Harry Nyquist
(Nyquist how many pulses to transfer from telegraph)
(two engineers from Bell labs)

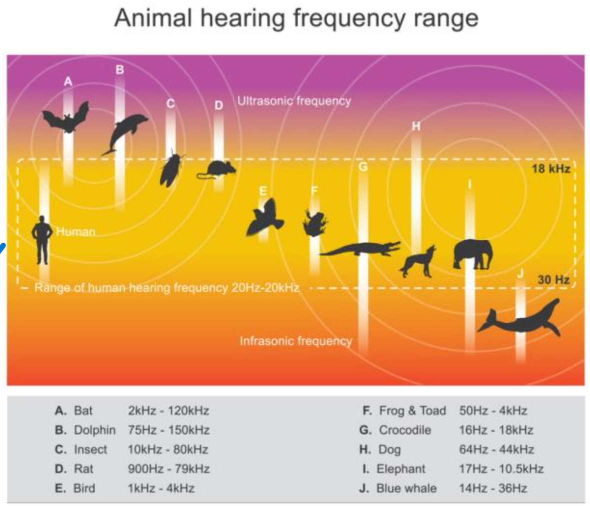
BANDWIDTH –LIMITED SIGNALS
אותות חסומי תדר

- Examples:

1. Hearing range 20 to 20,000 Hz
2. How preverbal infants see
3. Age-related vision, age related hearing





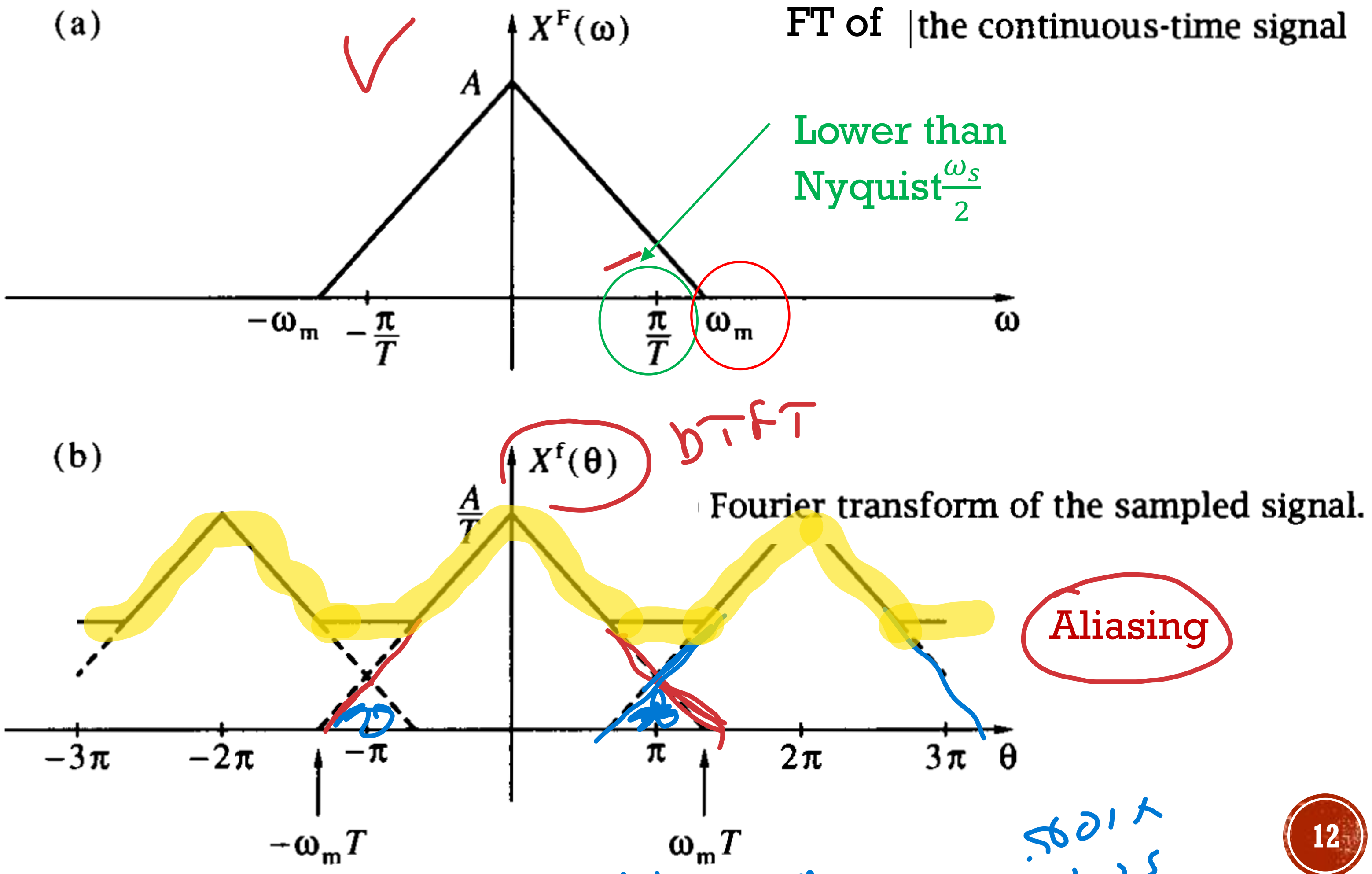


https://www.vectorstock.com/royalty-free-vector/hearing-range-describes-the-range-of-frequencies-vector-15444196

http://www.psy.vanderbilt.edu/courses/hon185/SpatialFrequency/SpatialFrequency.html

מקרה II

SAMPLING: CASE II - BANDWIDTH LIMITED SIGNAL SAMPLED AT LOWER THAN NYQUIST FREQUENCY



Clode Shannon and Harry Nyquist
(Nyquist how many pulses to transfer from telegraph)
(two engineers from Bell labs)

$$X_s(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

Aliasing — לקיפול זו אוסף

SAMPLING RATE EXAMPLES II: A RAISED COSINE SIGNAL

- Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1/f_0 & |\omega| \le \omega_1 \\ \dfrac{0.5}{f_0}\left[1 + \cos\left(\dfrac{|\omega| - \omega_0}{2\alpha f_2}\right)\right] & \omega_1 \le |\omega| \le \omega_2 \\ 0 & |\omega| > \omega_2 \end{cases}$$

raised cosine

$f_0 = 1/T$

Where $0 < \alpha < 1$ and

$\omega_1 = (1-\alpha)\pi f_0,\ \omega_2 = (1+\alpha)\pi f_0$

$x(nT) = x[n] = \delta[n]$-> DTFT$\{\delta[n]\} =$ const

תרגיל בית : ניתוח בזמן

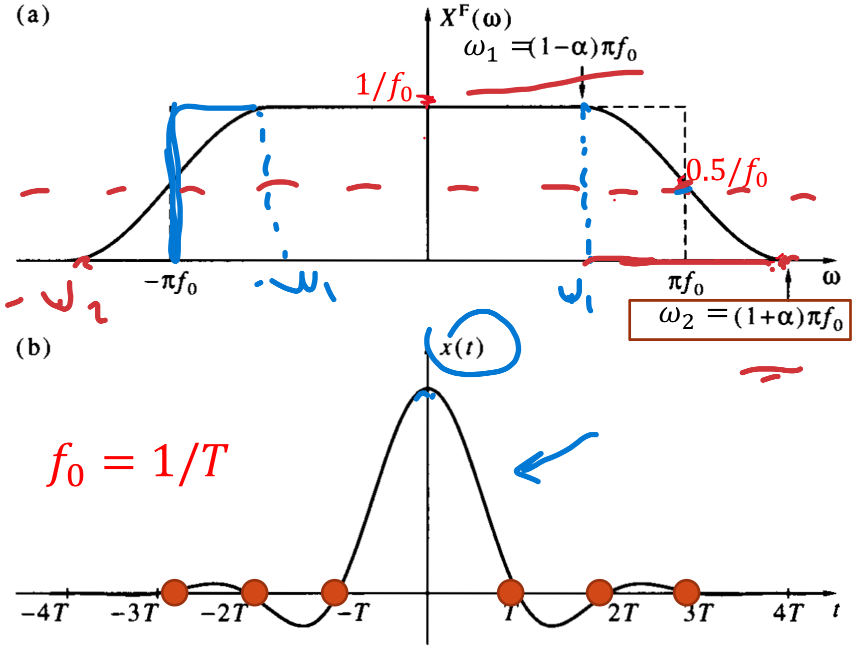
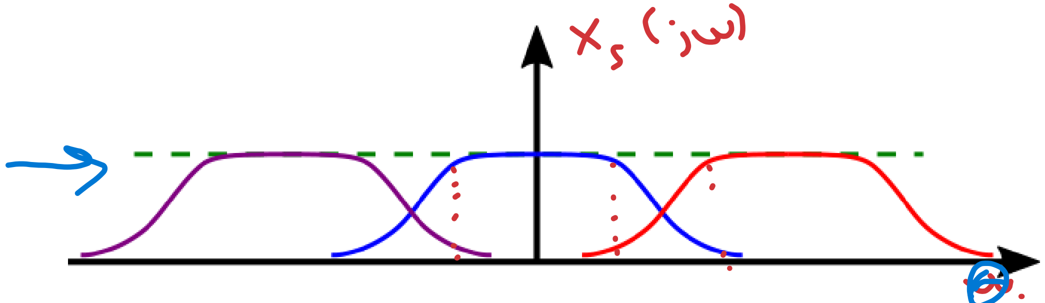


Figure 3.6 A raised-cosine signal: (a) the spectrum; (b) the waveform.



SAMPLING: CASE III – NOT BANDWIDTH LIMITED SIGNAL SAMPLING CONDITION IS NOT FULFILLED

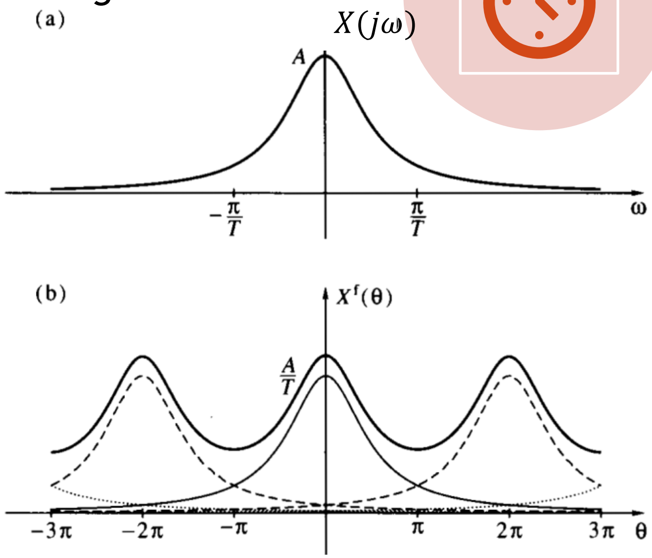
- Signals in nature are:
- Time limited
- Not bandwidth limited
- Cannot be sampled without error causing aliasing
- Sampling with minimal error

- Aliasing can't be avoided
- All the elements in the sum of replications will contribute to the aliasing effect

$$X^{f}(\theta) = X_{p}^{F}\left(\frac{\theta}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{F}\left(\frac{\theta - 2\pi k}{T}\right).$$

Aliasing

In this case the sum includes infinite number of nonzero terms so the shape of FT of the sampled signal must be distorted.



Conclusion: since signal in nature are time limited, they are not bandwidth limited, so sampling of infinite bandwidth signal always gives rise to aliasing, no matter how high the sampling rate is

משמעות – אותות אמיתיים
1) מוגבלי זמן
2) אינו מוגבל סרט
3) לא ניתן לדגום ללא אליאסינג
4) נדגום עם שגיאה מינימאלית

תכונה – אות מוגבל בזמן <- אינו מוגבל סרט
הוכחה – אות מוגבל זמן <- הכפלה בחלון <-
קונבולוציה ע"י סינק <- לא מוגבל סרט

SURVEY: CONVOLUTION

- EasyPolls:

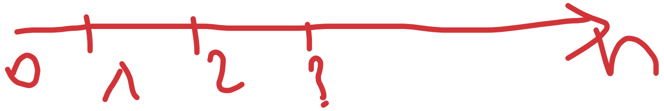
Delay by M samples can be realized by convolution with delta[n-M]. Can a delay by M+0.5 samples be realized?

- Yes
- No
- Yes, with a non-causal filter
- Yes, with a non-invertible filter

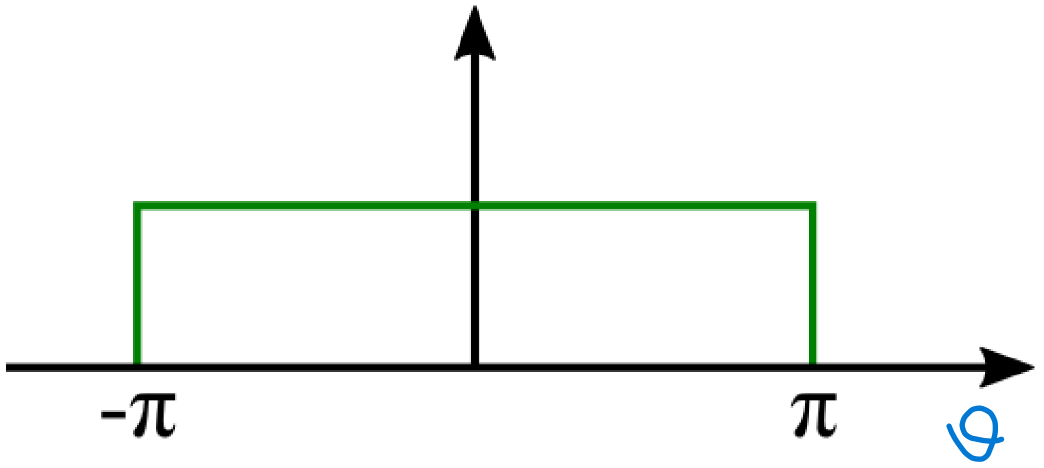
results vote

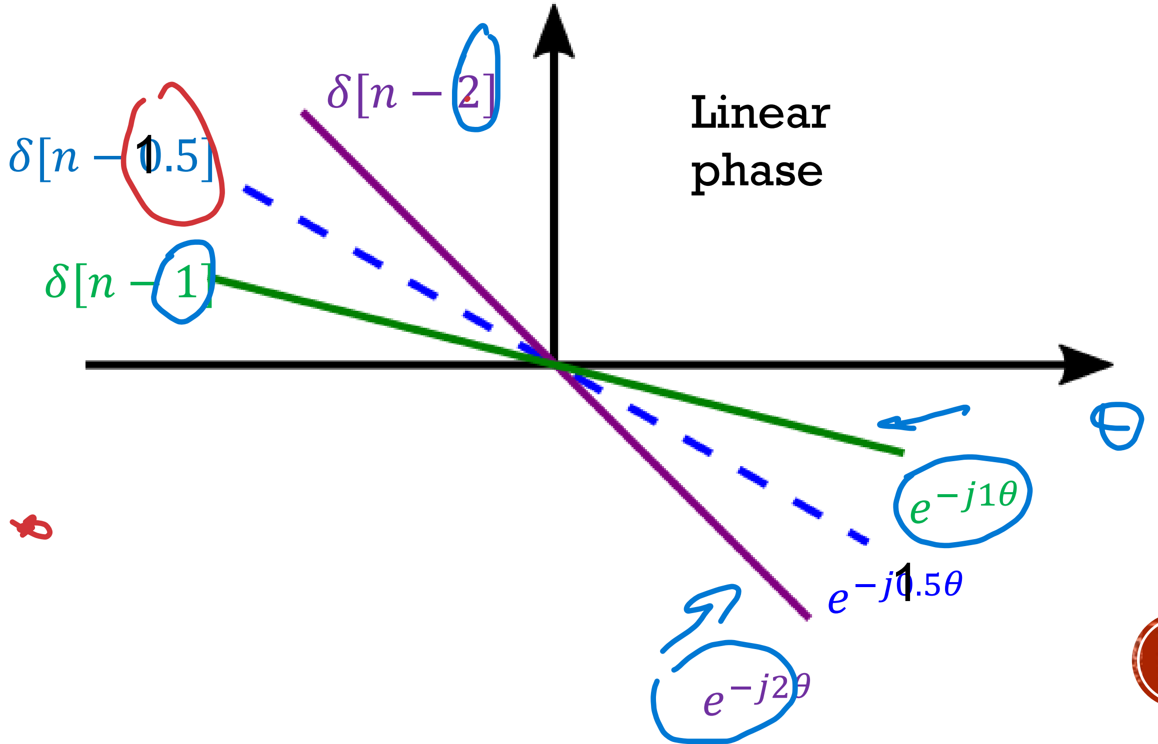
ראינו השהיה ב M דגימות
אשר ניתן לממש
באמצעות $\delta[n-M]$.
שאלה: האם ניתן לממש השהיה של $M+0.5$?

FRACTIONAL DELAY



Tricky in time, easy in frequency





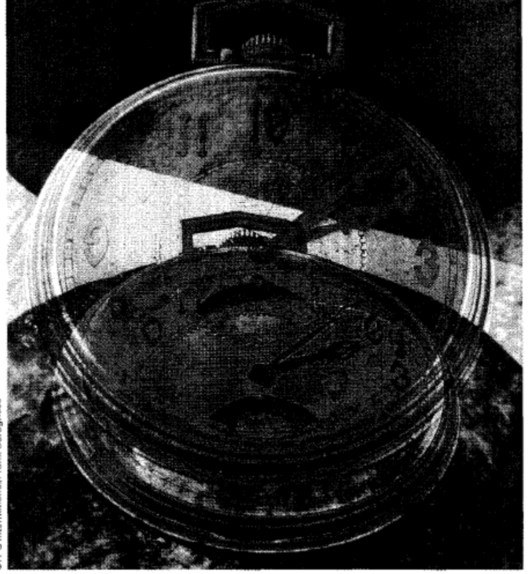
* בתדר - פשוט!

* בציר הזמן הבדיד כי
לא מוגדר ~~$\delta[n-0.5]$~~
$n \in \mathbb{Z}$

Splitting the Unit Delay

Tools for fractional delay filter design

TIMO I. LAAKSO, VESA VÄLIMÄKI,
MATTI KARJALAINEN, and UNTO K. LAINE



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A fractional delay filter is a device for bandlimited interpolation between samples. It finds applications in numerous fields of signal processing, including communications, array processing, speech processing, and music technology. In this article, we present a comprehensive review of FIR and allpass filter design techniques for bandlimited approximation of a fractional digital delay. Emphasis is on simple and efficient methods that are well suited for fast coefficient update or continuous control of the delay value. Various new approaches are proposed and several examples are provided to illustrate the performance of the methods. We also discuss the implementation complexity of the algorithms. We focus on four applications where fractional delay filters are needed: synchronization of digital modems, incommensurate sampling rate conversion, high-resolution pitch prediction, and sound synthesis of musical instruments.

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Overview

One fundamental advantage of digital signal processing techniques over traditional analog methods is the easy implementation of a constant delay: the signal samples are simply stored in a buffer memory for the given time. This technique works perfectly as long as the desired delay is a multiple of the used sample interval. However, when a delay of a fraction of the sample interval is needed or, particularly, if it is desired to control the delay value continuously, more sophisticated methods must be used.

The problem of implementing a fractional delay (FD) by digital means occurs in several applications. In one of the first treatments on the subject [23], a digital phase shifter was proposed for three problems: echo cancellation, phased-array antenna processing, and pitch-synchronous synthesis of speech. Later papers have included applications such as time delay estimation [88, 104], null steering in the direction pattern of antenna arrays [48, 101, 102], timing adjustment and interpolation in digital modems [29, 66, 67, 30, 31, 33, 32, 7, 27, 136, 10, 95, 96, 72], sampling rate conversion systems [3, 112, 55], stabilization of feedback systems [111], speech coding [50, 51, 68, 70, 78], speech-assisted video processing [18, 19], sub-pixel interpolation [20, 47], modeling of the human vocal tract [108, 58, 122, 125, 128, 131], and modeling of musical instruments [39, 42, 110, 120, 121, 123, 127]. A comprehensive study of modeling of acoustic tubes using fractional delay filters has been presented in [126] and [135]. There are many potential applications in video processing, such as frame interpolation and sub-pixel interpolation [20].

As a consequence of the wide range of application areas, the results on approximation of a fractional delay are scattered in the literature and difficult to find. The proposed design techniques concentrate almost exclusively on finite-impulse-response (FIR) filters. Moreover, the approach in the majority of papers known to us is limited. The problem is viewed mainly as a time-domain interpolation problem, often leading to a modification of standard sampling rate conversion methods [23, 24, 2], or to the use of the traditional Lagrange interpolation technique [58, 66, 67, 37]. One of the few textbook treatments on FD filter design can be found in [93].

Here, we take a filter designer's point of view on the fractional delay problem. After formulating the (generally complex-valued) approximation problem in the frequency domain, we systematically review the well-established filter design theory and search for efficient solutions to this particular problem. Our article thus has a review nature and a serious effort has been made to find all the relevant literature addressing the problem. Besides giving a systematic presentation of previous results, we use the employed frequency-domain approach to give deeper insight to many methods and also to reveal useful design techniques that have previously not been considered for this problem.

Preliminaries

Notation and Concepts

Delaying a continuous-time signal $x_c(t)$ by an amount t_d is conceptually simple. A continuous-time ideal delay can be defined as a linear operator, L_c, which yields its output, $y_c(t)$ as

$$y_c(t) = L_c\{x_c(t)\} = x_c(t - t_D) \tag{1}$$

(Fig. 1a). Delaying of a uniformly sampled bandlimited (baseband) digital signal must be treated with care. When we simply convert Eq. 1 into discrete time by sampling at time instants $t = nT$, where n is an integer and T is the sampling interval, we obtain

$n \in \mathbb{Z}$

$$y(n) = L\{x(n)\} = x(n - D) \tag{2}$$

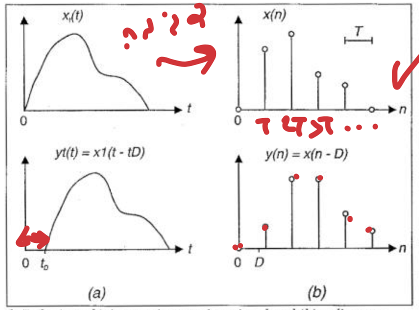
where D is a *positive real number* that can be split into the integer and fractional part as

$$D = \mathrm{Int}(D) + d \tag{3}$$

However, Eq. 2 is meaningful only for integer values of

Glossary of Notation

a_k	allpass filter coefficients	$h_{id}(n)$	impulse response of the ideal interpolator	$w(n)$	time-domain window function
$\mathbf{a}$	vector of allpass filter coefficients	$\mathbf{h}$	vector of FIR filter coefficients	$\mathbf{W}$	diagonal matrix with the elements of the window function $w(n)$
$A(z)$	allpass transfer function	$H(z)$	FIR transfer function	$W(\omega)$	frequency-domain weighting function
$c(n)$	coefficients of polynomial approximation	$H_{id}(z)$	the ideal transfer function	$x(n)$	filter input
$\mathbf{c}$	vector of cosine functions	L	FIR filter length ($L = N + 1$)	$y(n)$	filter output
$C(z)$	transfer function of polynomial approximation	n	discrete time index	α	passband width parameter ($0 < \alpha \leq 1$)
d	fractional part of the total delay D	N	filter order	$\beta(\omega)$	phase error function in allpass filter design
D	total noninteger delay to be approximated	M	net delay of the delay filter	δ	ripple parameter
$D(z)$	denominator of $A(z)$	P	order of spline or polynomial	$\Theta(\omega)$	phase response
$e(n)$	time-varying error sequence	$\mathbf{p}, \mathbf{P}$	vector and matrix in normal equations	Λ	diagonal matrix in group delay expression
$\mathbf{e}$	vector of complex exponentials	q	iteration index	$\tau_g(\omega)$	group delay response
E	general error measure	Q	number of polyphase branches	$\tau_p(\omega)$	phase delay response
$E(e^{j\omega})$	frequency-domain error function (complex-valued)	$r_{xx}(k)$	autocorrelation sequence of signal $x(n)$	ω	normalized angular frequency
f	frequency variable	$\mathbf{s}$	vector of sine functions	ω_p	passband cutoff angular frequency
$\mathbf{G}$	matrix in group delay expression	$\mathbf{S}$	matrix of sine or sinc functions	ω_s	stopband cutoff angular frequency
$h(n)$	impulse response of an FIR filter	$S_{xx}(\omega)$	power spectrum of signal $x(n)$	Ω_k	set of values of ω
		T	sampling interval		
		$\mathbf{V}$	Vandermonde matrix		



1. Delaying of (a) a continuous-time signal and (b) a discrete-time signal.

D. In that case, the output sample is one of the previous signal samples, but for noninteger values of D, the output value would lie somewhere between two samples, which is impossible (Fig. 1b). Instead, the appropriate values on the sampling grid must be found via *bandlimited interpolation* (the problem of approximating negative values of D calls for *extrapolation* or *prediction* algorithms, which are beyond the scope of this article).

The problem can be solved by viewing a delay as a *resampling process*. The desired solution can be obtained by first reconstructing the continuous bandlimited signal and then resampling it after shifting [30, 31]. The task is thus related to interpolation in multirate filter design techniques [23, 24, 118, 119] or sampling rate conversion in general [5, 6, 9, 91, 92, 112, 55]. Note, however that our basic constraint is to keep the sampling rate unchanged.

As in multirate applications, we need not perform reconstruction and resampling explicitly, but they can be reduced to appropriate linear filtering operating at the chosen sampling rate. A key issue is how to formulate the problem such that well-advanced filter design theory can be utilized in an efficient manner.

Like any linear time-invariant operation, delaying can equivalently be considered in a suitable transform domain. The z-domain transfer function of the system of Fig. 2 is obtained formally as

$$H_{id}(z) = \frac{Y(z)}{X(z)} = \frac{z^{-D}X(z)}{X(z)} = z^{-D} \qquad (4)$$

where $X(z) = Z\{s(n)\}$ and $Y(z)$ are the z-transforms of $x(n)$ and $y(n)$, respectively, and the subscript 'id' will stand for the desired (ideal) response. In Eq. 4, we have employed the following property of the z-transform:

$$Z\{x(n-D)\} = z^{-D}X(z) \qquad (5)$$

which, strictly speaking, holds for integer values of D only. The term z^{-D} represents precisely the ideal filter of Fig. 2 in the z-domain, which performs the desired bandlimited delay operation at the used sampling rate. Clearly, z^{-D} cannot be realized exactly for noninteger D, but it must be approximated in some way.

One approach for approximation is to construct a series expansion for z^{-D} as proposed in [90]. However, a more general and fruitful approach is to formulate the design objective in the frequency domain. In many applications the specifications are easier to give in the frequency domain, and numerous design techniques are available. The frequency response (Fourier transform) of the delaying system of Fig. 2 is obtained from Eq. 4 by setting $z = e^{j\omega}$

$$H_{id}(e^{j\omega}) = e^{-j\omega D} \qquad (6)$$

where $\omega = 2\pi fT$ is the normalized angular frequency, and T is the sample interval. The desired frequency response is thus a complex-valued function that specifies both the magnitude and the phase response as

$$|H_{id}(e^{j\omega})| \equiv 1 \text{ for all } \omega \qquad (7)$$

$$\arg\{H_{id}(e^{j\omega})\} = \Theta_{id}(\omega) = -D\omega \qquad (8)$$

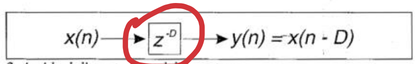
respectively. The phase information is often represented in the form of *group delay*, defined as the negative frequency derivative of the phase

$$\tau_g(\omega) = -\frac{\partial\Theta(\omega)}{\partial\omega} \qquad (9a)$$

or via *phase delay*

$$\tau_p(\omega) = -\frac{\Theta(\omega)}{\omega} \qquad (9b)$$

(see [81] or [84]).



2. An ideal discrete-time delay system.

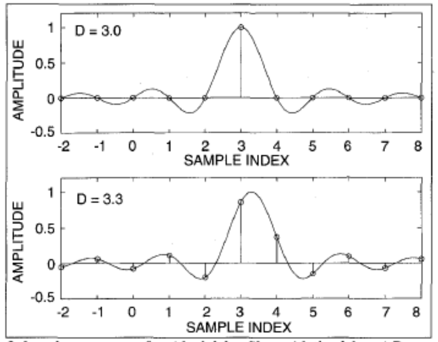
Both the group delay and the phase delay are measures for the delay of the system. Their difference is typically illustrated by considering an amplitude modulated signal for which the phase delay tells the delay of the carrier signal and the group delay tells that of the envelope (baseband signal) [81]. When the phase is exactly linear, the two delay measures yield identical results. For the frequency response Eq. 6, both the group delay and the phase delay are equal to the constant value D in the whole frequency band, i.e.,

$$\tau_{p,id}(\omega) = D \qquad (10a)$$

$$\tau_{g,id}(\omega) = D \qquad (10b)$$

Which of the three delay measures—phase, group delay, or phase delay—should be used? The physical viewpoint suggests the use of the phase or phase delay when the delaying of individual sinusoidal components of the signal is of interest. The group delay and the phase delay give flat curves for linear or almost-linear systems, which is more illustrative than the often steep-slope phase curve. As we are primarily interested in the delay of sinusoidal components, we have chosen to use the phase delay in the plots of this article. However, when introducing approximation methods, the phase or the group delay are also employed.

To summarize our discussion, we can use the ideal z-domain transfer function Eq. 4 or the ideal frequency response Eq. 6 as the design objective. These functions are complex-valued so that both the real and imaginary parts must be taken into account, separately or together. Alternatively, we can express this information in terms of magnitude and phase where the phase can be replaced by the group delay or the phase delay.



3. Impulse response of an ideal delay filter with the delay a) D = 3 and b) D = 3.3.

Ideal Solution

Assuming that the (real-valued) discrete-time signal represents a bandlimited baseband signal, the implementation of a constant delay can be considered as an approximation of the ideal discrete-time linear-phase allpass filter with unity magnitude and constant group delay of the given value D. The corresponding impulse response is obtained via the inverse discrete-time Fourier transform [84]:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \text{ for all } n \tag{11}$$

Substitution of Eq. 6 into Eq. 11 yields the solution for the ideal impulse response as

$$h_{id}(n) = \frac{\sin[\pi(n-D)]}{\pi(n-D)} = \mathrm{sinc}(n-D) \text{ for all } n \tag{12}$$

which has the shape of the familiar sinc function defined as

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{13}$$

When the desired delay D assumes an integer value, the impulse response Eq. 12 reduces to a single impulse at $n = D$, but for noninteger values of D the impulse response is an infinitely long, shifted and sampled version of the sinc function (Fig. 3). Unfortunately, the ideal impulse response is not only infinitely long but also noncausal, which makes it impossible to implement it in real-time applications.

Equation 12 gives an answer to the original problem, i.e., where the delayed signal value should be placed as it cannot be put "between the samples." In the ideal case, it is to be spread over all the discrete-time signal values, weighted by appropriate values of the sinc function.

This simple result (Eq. 12) is of fundamental importance since, whatever method is used, the impulse response of the approximating (real-coefficient) filter must imitate this ideal response in some meaningful sense. It is also evident that with a finite-order causal FIR or IIR filter the ideal response can only be approximated. Furthermore, the ideal solution from Eq. 12 can be utilized in the formulation of the approximation problem in the time domain, as will be discussed in more detail in the following sections.

Fractional Delay Approximation Using FIR Filters

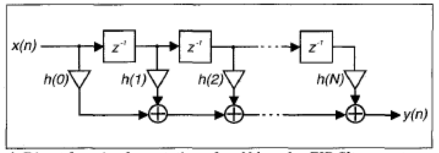
Let us consider the approximation of the ideal fractional delay D by an Nth-order (length $L = N + 1$) FIR filter with the z-domain transfer function

$$H(z) = \sum_{n=0}^{N} h(n)z^{-n} \tag{14}$$

An FIR filter is typically implemented with the direct form structure of Fig. 4. It is now desired to determine the coefficients $h(n)$ such that the chosen norm of the frequency-domain error function

$$E(e^{j\omega}) = H(e^{j\omega}) - H_{id}(e^{j\omega}) \tag{15}$$

is minimized. Note that both the desired function $H_{id}(e^{j\omega})$ and the error function are *complex-valued*. This complicates the solution in general as compared to a real-valued approximation problem in linear-phase FIR filter design [84]. However, simple solutions can still be found as will be shown in the following example.



4. Direct form implementation of an Nth-order FIR filter.

IDEAL RECONSTRUCTION

נתון אות מוגבל סרט $x(t)$ בפס רוחב סרט ω_m.

האות נדגם לפי קצב נייקוויסט כנדרש $f_s \geq 2f_m$ לקבלת $x[n]$

שאלה: איך משחזרים $x(t)$ בהינתן $x(nT)$

RECONSTRUCTION: BAND LIMITED X(T)

Theorem 3.2 A band-limited signal $x(t)$ whose bandwidth is smaller than $\pm\pi/T$ can be exactly reconstructed from its samples $\{x(nT),\ -\infty < n < \infty\}$, using the formula

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right).$$

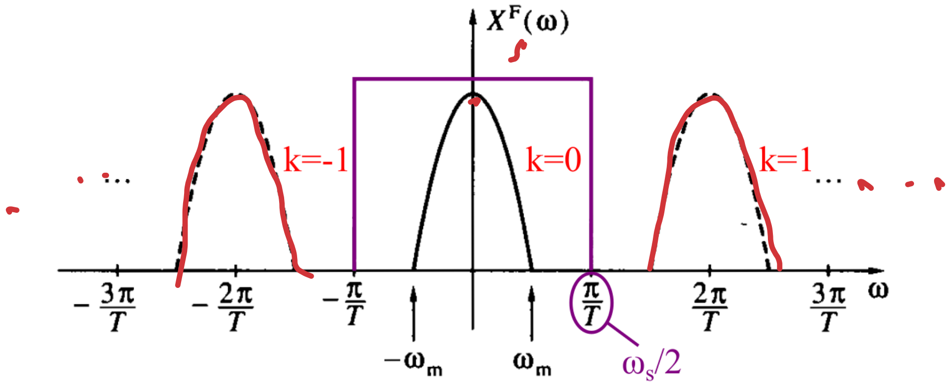


Figure 3.10 Reconstruction of a band-limited signal by an ideal low-pass filter (replicas shown by dashed lines will be eliminated by the filter).

Proof: $\boxed{X^F(\omega) = T X_p^F(\omega) H^F(\omega)}$

where $H^F(\omega)$ is an ideal low-pass filter, that is,

$$H^F(\omega) = \mathrm{rect}\left(\frac{\omega T}{2\pi}\right)$$

Therefore, $x(t)$ is given by the convolution

$$x(t) = T\int_{-\infty}^{\infty} h(\tau)x_p(t-\tau)\,d\tau.$$

The impulse response of the ideal low-pass filter is

$$h(t) = \frac{1}{T}\mathrm{sinc}\left(\frac{t}{T}\right)$$

Therefore,

$x(t)=h(t)*x_s(t)$ ⟶

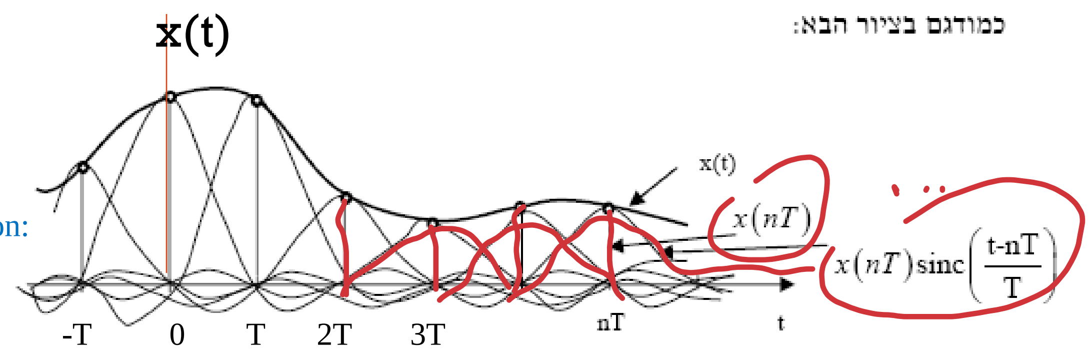
$$x(t) = \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{\tau}{T}\right)\left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-\tau-nT)\right]d\tau \overset{\text{sifting}}{=} \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

Shannon reconstruction

EXAMPLE RECONSTRUCTION OF BAND LIMITED X(T) VIA SINC

עפ"י הבטוי שהתקבל לעיל, האינטרפולציה מתבצעת על ידי סכום משוקלל של פונקציות sinc מוזזות, כמודגם בציור הבא:

Reconstruction of a continuous-time signal using ideal interpolation:



שחזור אות רציף מדגימותיו נקרא אינטרפולציה. עלינו להניח כי:

- אות מקורי היה חסום בסרט כך ש B Hz < $0.5fs$
- תמיד להשתמש בחצי תדר נייקויסט! $0.5fs$

הערה: מסנן **אינטרפולציה האידיאלי** הוא מהצורה: $H(\omega) = T\,\Pi\left(\frac{\omega}{2\pi f_s}\right) = T\Pi\left(\frac{\omega}{\omega_s}\right)$

כאשר עבור אינטרפולציה מוגבלת סרט של Shannon: $h_r(t) = \text{sinc}\left(\frac{t}{T}\right)$

נוסחת אינטרפולציה בצורתה הכללית: $x_r(t) = x(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_r(t-nT)$

$h_r(t)$- נקראת *גרעין האינטרפולציה* (Interpolation kernel).

גרעין אינטרפולציה

סינון מוגבל סרט, יכול לשחזר ספקטרום מקורי:

RECONSTRUCTION

(Sampling Theorem) משפט הדגימה

יהיה $x(t)$ אות רציף בזמן בעל התמרת פוריה $X^F(\omega)$ מוגבלת סרט, דהיינו $X^F(\omega)=0$ עבור $|\omega|>\omega_m$. אזי, ניתן לשחזר את ערכי $x(t)$ בכל רגע t שהוא מתוך ערכי דגימותיו, $x(nT)$, $T<\frac{\pi}{\omega_m}$, על ידי הביטוי:

Reconstruction equation of Shannon

$$\boxed{x(t)=\sum_{n=-\infty}^{\infty} x(nT)\underbrace{\operatorname{sinc}\left(\frac{t-nT}{T}\right)}_{\text{reconstruction function}}}$$

sifting

כאשר

$$\operatorname{sinc}(t)\triangleq\frac{\sin(\pi t)}{\pi t}$$

$$\underline{x(t)} = h(t)*x_s(t) =$$

$$= \int_{-\infty}^{\infty} \operatorname{sinc}(r)\cdot\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT-\tau)\,d\tau =$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}\left(\frac{t-nT}{T}\right)$$

הערה: אם ל- $X^F(\omega)$ אין פ' דלתה ב- $\omega=\pm\omega_m$ וכן קיים $X^F(\omega_m)=X^F(-\omega_m)$, אפשר להשתמש ב- $T=\frac{\pi}{\omega_m}$ (כלומר $\omega_s=2\omega_m$). כמובן שהדבר כך אם $X^F(\omega_m)=0$.

הערה: $h_r(t)=sinc\left(\frac{t}{T}\right)$ גרעין אינטרפולציה אידיאלית!

RECONSTRUCTION: BAND LIMITED X(T)

When is the sampling condition is Not fulfilled?

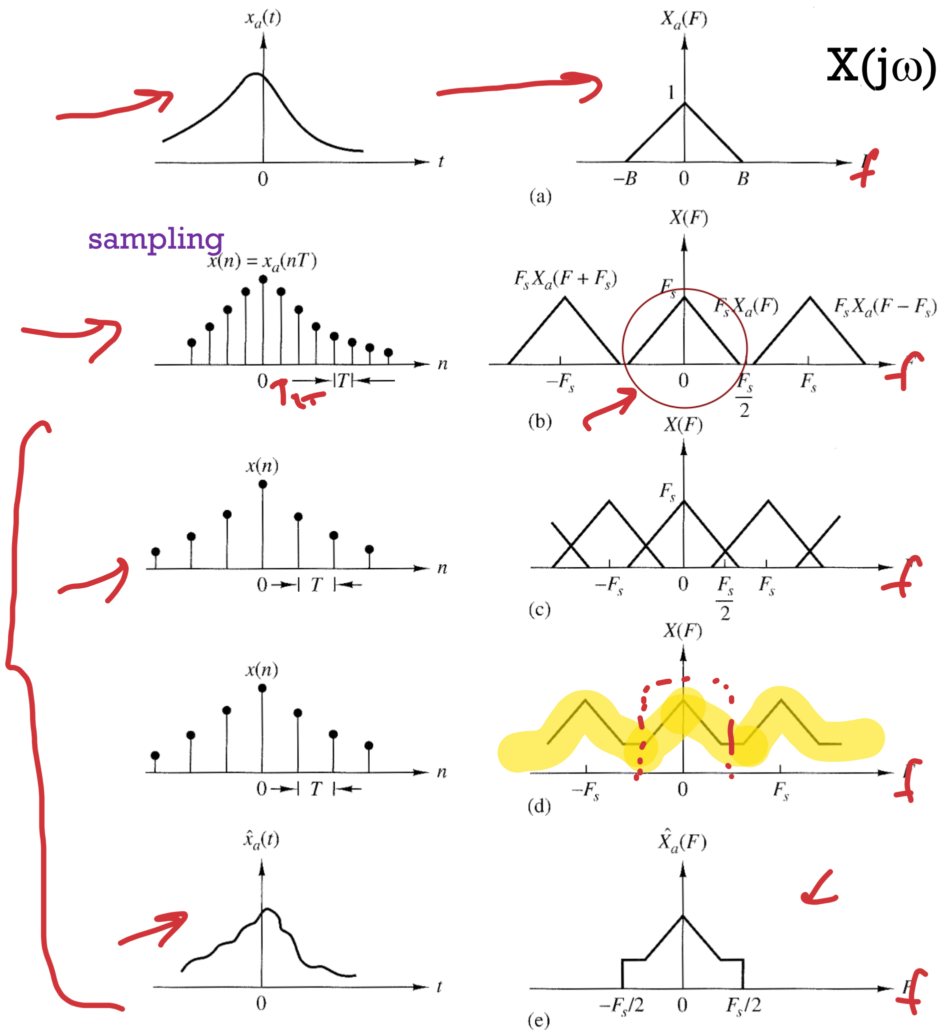


Figure 6.1.1 Sampling of an analog bandlimited signal and aliasing of spectral components.

1

2 replications

3 Overlapping replications

4 Aliasing

5

SURVEY: RECONSTRUCTION

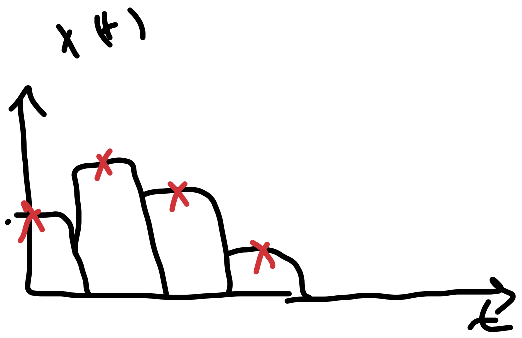
- EasyPolls:

Is there a signal that can be fully reconstructed from its samples via a time-limited reconstruction function

- ○ Such a signal does not exist
- ○ Such a signal does exist but it is bandwidth limited
- ○ Such a signal does exist but it is not bandwidth limited ✓

results | vote

הוא קיים כל
אשר ניתן לתאר
מקטעים קבועים
סופי, שימנים
מרובע מעבר ?



$$\sum_{-\infty}^{\infty} x(nT)\cdot \Pi\left[\frac{t-nT}{T}\right]$$

The impulse response of the ZOH device is

$$h_{\mathrm{zoh}}(t) = \begin{cases} 1, & 0 \le t < T, \\ 0, & \text{otherwise.} \end{cases} \quad (3.33)$$

The impulse response (3.33) is shown in Figure 3.11; the response $\hat{x}(t)$ to the input sequence $x(nT)$ is shown in Figure 3.12. The latter figure also shows, in a dashed line, the ideal waveform $x(t)$, which would be obtained by a Shannon reconstructor.

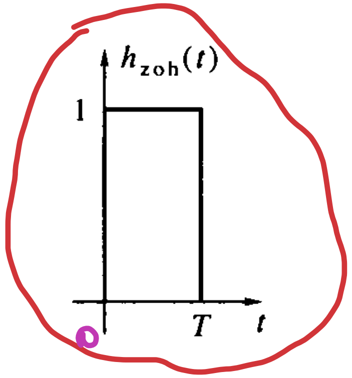


Figure 3.11 Impulse response of a zero-order hold.

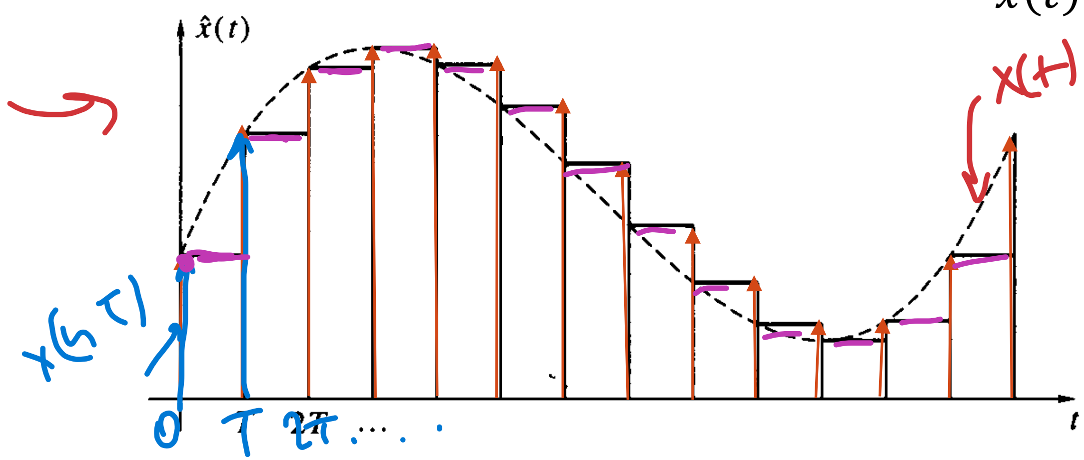


Figure 3.12 Time response of a zero-order hold. Staircase line: actual response $\hat{x}(t)$; dashed line: ideal response $x(t)$.

Zero-order hold (ZOH)
שחזור מסדר אפס

- השיחזור לפי שנון דורש אינסוף דגימות אחורה וקדימה, ולכן שיחזור שנון אינו מעשי.

$\hat{x}(t) = x_s(t) * h(t)$

n≠0 ∀ nT≤ t

n≤t/T

due to causality

- משחזר מעשי – יהיה סיבתי: $\hat{x}(t) = \sum_{n=-\infty}^{t/T} x(nT)h(t-nT)$
- במשחזר טוב $\hat{x}(t)$ דומה ל $x(t)$
- משחזר פשוט: ZOH $\hat{x}(t) = x[n], nT \le t < (n+1)T, n \in \mathbb{Z}$

Question: what is the frequency response of the $H\{h_{ZOH}(t)\}$?

The frequency response of the H{$h_{ZOH}(t)$}

To understand to what extent $\hat{x}(t)$ approximates $x(t)$, let us compute the frequency response of the ZOH. As we recall, the frequency response of the ideal (Shannon) reconstructor is a perfect rectangle on $-\pi/T \le \omega \le \pi/T$, with zero phase; see (3.25). By analyzing how the frequency response of the ZOH deviates from the perfect rectangle, we will understand the nature of distortions introduced by the ZOH. We have from (3.33)

Euler=$\sin(z)=\frac{e^{jz}-e^{-jz}}{2j}$

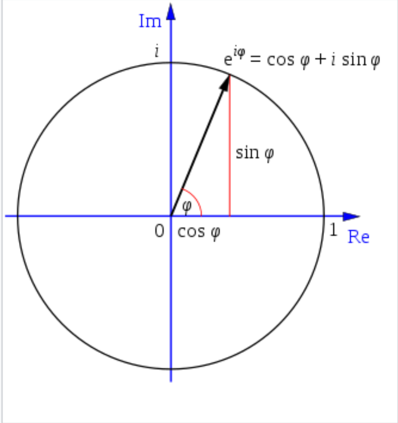
$$\hat{x}(t) = x_s(t) * h(t) \longrightarrow \hat{X}(j\omega) = X_s(j\omega)\cdot H(j\omega)$$

$$FT\{h(t)\} = H(j\omega) = \int_0^T 1\cdot e^{-j\omega t}\,dt = \frac{1-e^{-j\omega T}}{j\omega} = T\cdot e^{-j0.5\omega T}\cdot\frac{e^{j0.5\omega T}-e^{-j0.5\omega T}}{2\cdot 0.5\,j\omega T} = T\,e^{-j0.5\omega T}\,\frac{\sin(0.5\omega T)}{0.5\omega T} =$$

$$= T\,e^{-j0.5\omega T}\,\frac{\sin(\pi\omega T/\pi 2)}{\pi\omega T/\pi 2} = T\cdot\mathrm{sinc}\left(\frac{\omega}{\omega_s}\right)\cdot e^{-j0.5\omega T} = T\,sinc\left(\frac{\omega T}{2\pi}\right)e^{-j0.5\omega T}$$

$\omega_s=\frac{2\pi}{T}$

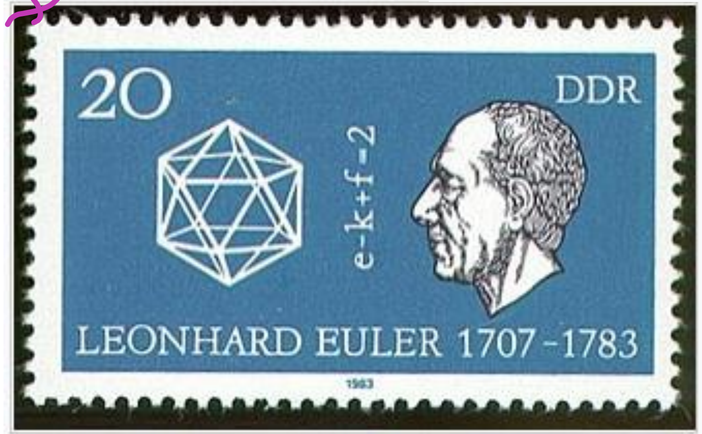
LEONHARD EULER — HISTORICAL NOTE



A geometric interpretation of Euler's formula

Saint Petersburg

1957 Soviet Union stamp commemorating the 250th birthday of Euler. The text says: 250 years from the birth of the great mathematician, academician Leonhard Euler.



Stamp of the former German Democratic Republic honoring Euler on the 200th anniversary of his death. Across the centre it shows his polyhedral formula, in English written as "$v - e + f = 2$".

Leonhard Euler

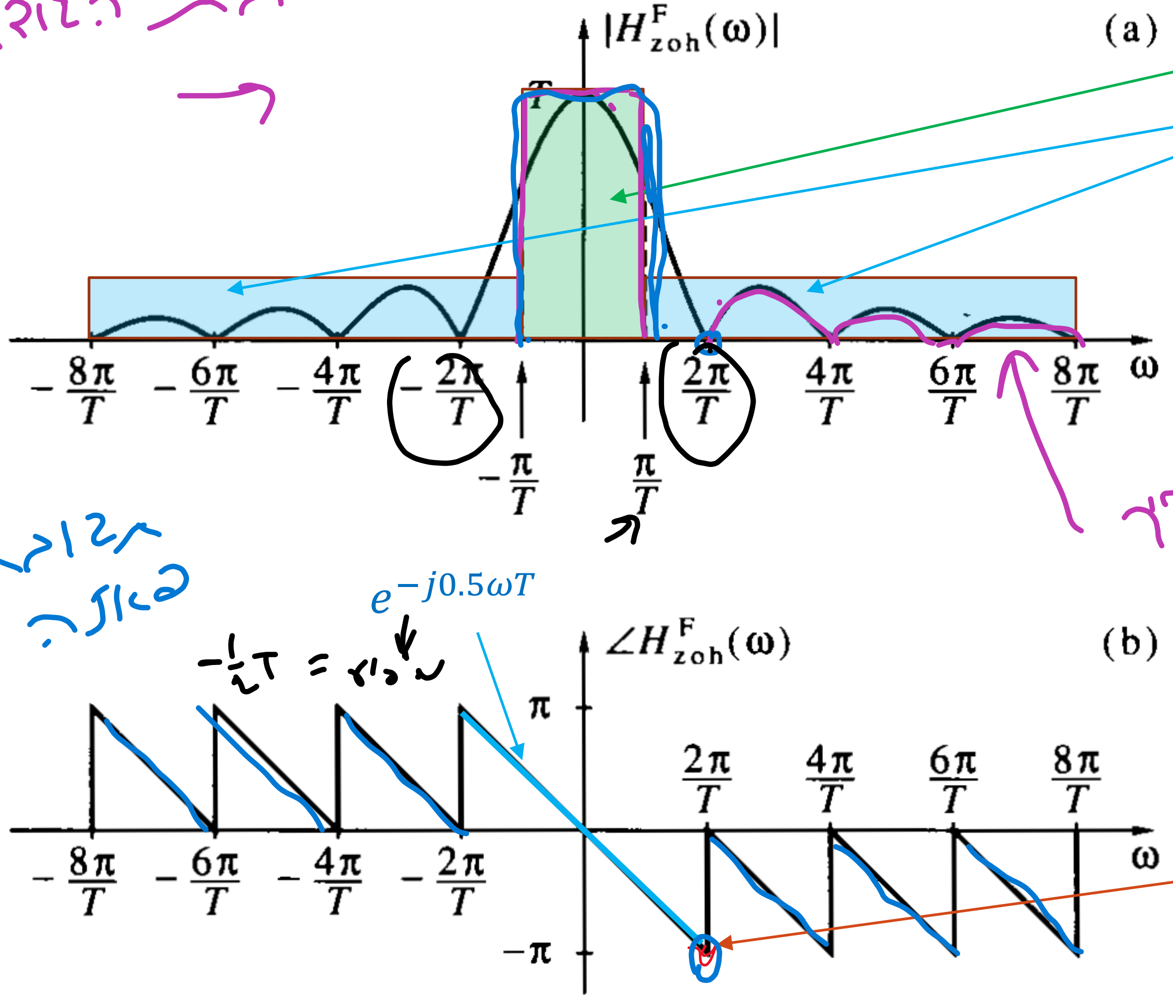
Portrait by Jakob Emanuel Handmann (1753)

Born	15 April 1707 Basel, Switzerland
Died	18 September 1783 (aged 76) [OS: 7 September 1783] Saint Petersburg, Russian Empire
Alma mater	University of Basel (MPhil)

Euler

RECONSTRUCTION

תגובת הגבר / מגניטודה



תגובת הפאזה

Figure 3.13 Frequency response of a zero-order hold: (a) magnitude; (b) phase.

- בתחום $|\omega| \le \omega_s/2$ הגבר קטן או שווה ל T
- בתחום $|\omega| > \omega_s/2$ הגבר שונה מ 0
- תדרים גבוהים – שינוי חד בזמן (מדרגות)
- משמעות : שכפולים לא מתאפסים – רק מונחתים

נניח
אם רוצים
סינק.
אין לנו יכולת
לממש אותו

- פאזה – סינק ב ± 1

DIGITAL-TO-ANALOG CONVERSION

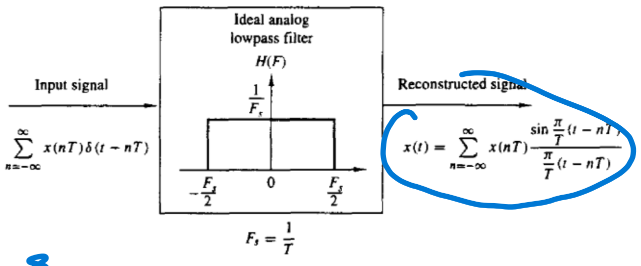


Figure 9.18 Signal reconstruction viewed as a filtering process.

MAGNITUDE AND PHASE COMPARISON TO IDEAL LOW-PASS FILTER

1. הגבר: זהה ושווה ל-T בתחום $|\omega| \le \omega_s/2$.
2. הפאזה: שונה מ-0 בתחום $|\omega| > \omega_s/2$.
3. מדגמים בזמנים ← שינוי חד בתדר (מעברים).
4. משמעות: מסננים אלו מתאימים רק מותאמים.
5. כיצד משפרים? ___ סינון נוסף (עם T 0.5 - שינוי בתחום

$|\omega| \le \omega_s/2$

MAGNITUDE AND PHASE COMPARISON TO IDEAL LOW-PASS FILTER

The magnitude and phase responses of $H_{\text{zoh}}^{\text{F}}(\omega)$ are shown in Figure 3.13. We observe the following differences with respect to the ideal low-pass filter:

1. The magnitude response at low frequencies is not flat, but decays gradually. Furthermore, it decreases to zero at $\omega = \pm 2\pi/T$, rather than at $\pm\pi/T$.
2. The magnitude response has nonvanishing ripple at high frequencies, so the reconstructed signal $\hat{x}(t)$ has undesired high-frequency energy. In the time domain, the high-frequency energy is apparent in the staircaselike form of the output created by the hold operation.
3. The phase of the response is not zero, but piecewise linear, with slope $-0.5T$.